An Approximating-Interpolatory Subdivision Scheme.

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Abstract

In the last decade, study and construction of quad/triangle subdivision schemes have attracted attention.

The quad/triangle subdivision starts with a control mesh consisting of both quads and triangles and produces finer and finer meshes with quads and triangles (Fig. 1). Designers often want to model certain regions with quad meshes and others with triangle meshes to get better visual quality of subdivision surfaces. Smoothness analysis tools exist for regular quad/triangle vertices. Moreover C^1 and C^2 quad/triangle schemes (for regular vertices) have been constructed. But to our knowledge, there are no quad/triangle schemes that unifies approximating and interpolatory subdivision schemes.

In this paper we introduce a new subdivision operator that unifies triangular and quadrilateral subdivision schemes. Our new scheme is a generalization of the well known Catmull-Clark and Butterfly subdivision algorithms. We show that in the regular case along the quad/triangle boundary where vertices are shared by two adjacent quads and three adjacent triangles our scheme is C^2 everywhere except for ordinary Butterfly where our scheme is C^1.

AMS Subject Classification: Numerical analysis, Computer science.

Key Words and Phrases: Subdivision, Polynomial generation, Quad/triangle Subdivision, Quasi-interpolants.

1 Introduction

Since their first appearance in 1978, subdivision algorithms for generating surfaces of arbitrary topology have gained widespread popularity in computer graphics and are being evaluated in engineering applications. This development was complemented by ongoing efforts to generate appropriate mathematical tools for a thorough analysis, and today, many of the fascinating properties of subdivision are well understood. Subdivision surfaces were introduced in 1978 by both Catmull and Clark [7] and Doo and Sabin [9]. They both generalized tensor product B-spline of bi-degree three and two respectively to arbitrary topologies by extending the refinement rules to irregular parts of the control mesh. Later, in 1987, Loop studied behavior of recursive subdivision surfaces near extraordinary points and generalized triangular box-splines of total degree four to arbitrary triangular meshes in [10].

The visual quality of a subdivision surface is highly dependent on the initial mesh of control points. For general shapes, designers often want to model some areas with triangular patches, and others with quad patches. So to combine advantages of both surfaces, it is often desirable to have surfaces that have a combined quad-triangle patch structure.

In 2003, Stam and Loop [6] introduced a generalization of the Loop and Catmull-Clark subdivision that unifies these schemes and operate on mixed quad/triangle surfaces. The subdivision scheme reproduces Loop subdivision on triangular portions of the mesh and Catmull-Clark subdivision on quadrilateral polygons. The quad/triangle boundary is subdivided with an averaging mask of Catmull-Clark and Loop subdivision in the quad/triangle boundary. Stam and Loop showed that their scheme is C^2 everywhere except for extraordinary points and the ordinary quad/triangle boundary where their scheme is C^1. To remedy this smoothness problem along ordinary quad/triangle edges, Levin and Levin [5] introduced a set of modified rules along the quad/triangle boundary. The authors also presented the concept of an "unzippering" mask. Like Levin and Levin, Schaefer and Warren [8] used an unzippering mask during subdivision. Then the authors proved that these modified rules generate a surface that is C^2 across the quad/triangle boundary.

In 2009, Jiang et al [1] introduced an interpolatory quad/triangle subdivision scheme.

All the quad/triangle schemes available in the literature are only approximating schemes or only interpolatory schemes. In reverse engineering, 3D data points on surfaces often come from 3D optical measurement (for instance a 3D laser scanner). It may be useful to produce interpolatory schemes to fit 3D data with enough confidence and only approximating schemes on noisy data for instance. In CAGD, constraint may be imposed on parts of objects. These constraints may have a variable tolerance (from 0 for interpolatory constraints to ϵ for allowed geometric inaccuracies in relative locations and shapes of features giving raise to approximating constraints). In these approaches, any combination of interpolatory scheme (either on quads or triangles) and approximating scheme (again on quads or triangles) may be desired. We shall limit ourselves here on an approximating quad interpolatory triangle scheme.

To construct an approximating/interpolatory quad/triangle scheme, one first needs to choose an approximating scheme for quad vertices and choose an other interpolatory scheme for triangle vertices. The Catmull-Clark scheme and the Butterfly scheme in [4] and its modified version [11] are probably the most popular approximating and interpolatory schemes for quad vertices and for triangle vertices respectively.

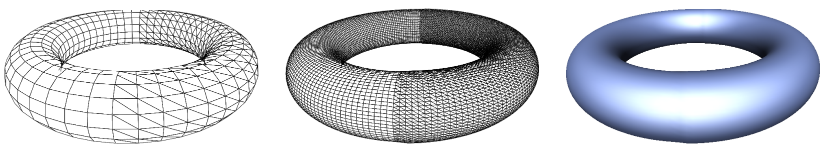

Figure 1: *A quad/triangle subdivision.*

The present paper is organized as follows. In Section 2, we give the formal definition of the quad/triangle subdivision schemes and some properties of quasi-interpolants are given. Then we present our method to construct a subdivision scheme that unifies an approximating (Catmull-Clark) and an interpolatory (Butterfly) subdivision schemes in the regular case. We explain how to calculate the weight stencils of our scheme in section 3. The eigenanalysis of our subdivision scheme is detailed in section 4. In section 5, we give some results of applications of our subdivision scheme. Finally, conclusion and future work are given in section 6.

2 Background

In this section we give the formal definition of the quad/triangle subdivision schemes, since the framework of quasi-interpolants and the theory of polynomial generation are particularly useful for combining two uniform stationary subdivision schemes along a common boundary.

2.1 Notations

Let $C = C^m(\mathbb{R}^s)$ denote the space of all m-times differentiable functions from $\mathbb{R}^s$ to $\mathbb{R}$ whose m-th order derivatives are continuous. The function space of polynomials of degree$\leq m$ over the domain $X \subset \mathbb{R}^s$ is:

$$\Pi_m(X) = span\{f | f : X \to \mathbb{R}, f(x) = x^i, 0 \leq |i| \leq m\}.$$

We refer to $P \in l$ as a set of control points, where $l = l(\mathbb{Z}^s)$ denotes the collection of all sequences $P : \mathbb{Z}^s \to \mathbb{R}$. For the identity operator $Id : X \subset \mathbb{R}^s \to \mathbb{R}^s$, we have $Id(x) = x, \forall x \in X$

Let $j = (j_1, j_2, ...j_s) \in \mathbb{Z}^s$, and $x \in \mathbb{R}^s$. Then in multi-index notation, we have:

$|j| = |j_1|+|j_2|+....+|j_s|$, $j \geq 0 \Leftrightarrow (j_1, j_2, ..., j_s) \geq (0, 0, ..., 0)$, $x^j = x_1^{j_1} x_2^{j_2} .....x_s^{j_s}$.

For $j \geq 0$, we use $D^j f = \frac{\partial^{|j|} f}{\partial^{j_1} x_1 .. \partial^{j_s} x_s} = f_{x_1 x_2 .. x_s}$.

Entries of a multi-index sequence $P \in l$ are denoted by $P_{(j)}$, for $j \in l(\mathbb{Z}^s)$, and $SP \in l$ by $(SP)_{(j)}$, where S is a subdivision operator. Let $\lambda \in \mathbb{R}^s$. For arithmetic operations on subsets of $X \subset \mathbb{R}^s$ we define:

$$\lambda X = \{\lambda x \mid x \in X\}$$

$$X + z = \{x + z | x \in X, z \in \mathbb{R}^s\}$$

We denote by σ the dilation operator on C, which appears naturally in the context of stationary subdivision:

$$\sigma f(.) = f(\frac{.}{2}).$$

2.2 Quad/triangle subdivision

The following framework is a specialization of the more general definition of non uniform subdivision introduced in [5]. We consider a quasi-uniform grid $X \in \mathbb{R}^2$, namely a grid which is uniform in each of the half planes, $X > 0$ and $X < 0$, and such that $\xi X \equiv \{(i, j+1)|(i,j) \in X\} = X$, $2X \subset X$ and $\cup_{n=0}^{\infty} 2^{-n} X = \mathbb{R}^2$ as shown in Fig. 2.

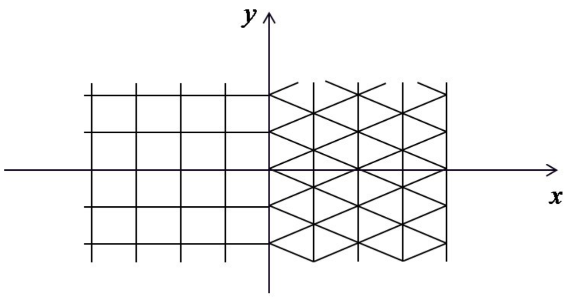

Figure 2: *A quad/triangle grid.*

Let $l(X)$ denote the space of all control point sequences $l(X) = \{P|P : X \to \mathbb{R}\}$. The subdivision operator S is a linear operator on $l(X), S : l(X) \to l(X)$. A stationary subdivision scheme is defined as the repeated application of S to the given control points $P \in l(X)$. S is said to be convergent, if for every $P \in l(X)$, there exists $F \in C(R^2)$ (called the limit function) such that

$$\lim_{n\to\infty} \|S^n P - F(2^{-n})\|_{\infty, X\cap 2^n D} = 0, \quad (1)$$

for any open and bounded domain $D \subset \mathbb{R}^2$. We denote $S^\infty P = F$. It is also required, as part of the definition of uniform convergence, that $S^\infty P$ is nonzero for some P. Notice that although $S^n P$ is formally defined as a sequence over X, the value $S^n P(x)$ for $x \in X$, is associated with the value of the limit function at $2^{-n}x$, as implied by (1). S is C^m if $S^\infty P \in C^m(\mathbb{R}^2)$ for any $P \in l(X)$. A quasi-uniform bivariate scheme consists of different uniform rules on each side of the y-axis, far enough from the axis, and of different rules near the y-axis. This scheme is uniform in the y-direction. S is assumed to be C^m continuous away from the y-axis, and the bivariate scheme is assumed to generate Π_m, the space of bivariate polynomials up to degree m. The last requirement implies the existence of an inverse Q of S^∞ on Π_m. Levin in [2] shows that every stationary uniformly convergent subdivision scheme satisfies

$$S^\infty(P) = \sum_{|i|\leq m} \frac{m_i}{i!} D^i f \quad \forall f \in \Pi_m(\mathbb{R}^s), \tag{2}$$

in which the moments m_i are given by:

$$m_i = \sum_{\beta\in\mathbb{Z}^s} \Phi(-\beta)\beta^i. \tag{3}$$

Equation (2) expresses the operator $S^\infty : \Pi_m(\mathbb{Z}^s) \to \Pi_m(\mathbb{R}^s)$ as a linear combination of differential operators. The operator $Q : \Pi_m(\mathbb{R}^s) \to \Pi_m(\mathbb{Z}^s)$ that satisfies

$$S^\infty Q = Id, \tag{4}$$

is called the quasi-interpolant of the scheme S.

Using equations (2),(3) and (4) the quasi-interpolants operators of cubic B-spline and Catmull-Clark subdivision schemes are given by equations (5) and (6):

$$Qf = f - \tfrac{1}{6} f_{xx} \quad \forall f \in \Pi_3. \tag{5}$$

$$Qf = f - \tfrac{1}{6}(f_{xx} + f_{yy}) \quad \forall f \in \Pi_2. \tag{6}$$

Note that the quasi-interpolant operator of an interpolatory scheme is given by:

$$Qf = f \quad \forall f \in \Pi_m. \tag{7}$$

The important properties of the quasi-interpolant operator Q are summarized in the following theorem of Levin [2].

Theorem 1. *If S is a convergent subdivision scheme, if S^∞ is an injection, and the quasi-interpolant operator Q preserves leading coefficients, then:*

$$SQf = Q\sigma f \Longleftrightarrow S^\infty Qf = f, \forall f \in \Pi_m. \tag{8}$$

Note that $Q : \Pi_m \to L(X)$ preserves leading coefficients if

f$\in \Pi_k \Rightarrow |Qf(X) - f(X)| = 0$ as $\|X\| \to \infty, x \in X$, (9)

for all $k \leq m$.

From this theorem we get important information about the eigenvalues and the eigenvectors of S. Considering a monomial $f = x^i y^j$, with $i + j \leq m$, it follows that $\sigma f = 2^{-(i+j)} f$ and thus:

$$SQ(x^i y^j) = Q\sigma(x^i y^j) = 2^{-(i+j)} Q(x^i y^j), i + j \leq m, \qquad (10)$$

where $Q(x^i y^j)$ is an eigenvector of the scheme for $i + j \leq m$ associated with the eigenvalue $2^{-(i+j)}$.

3 Approximating-Interpolatory Subdivision scheme

Considering the quad/triangle grid in Fig. 2, we would like to define a quasi-uniform scheme over this grid which coincides with the tensor product cubic B-spline scheme, or the Catmull-Clark subdivision scheme, on the left half-plane and the Butterfly subdivision scheme on the right half-plane.

The stencils of these two schemes in the regular case are given in Fig. 3.

The goal is to define special rules on the y-axis and near it. These special rules are constructed together with the quasi-interpolant operator Q, which also requires a special definition near the y-axis, so that the condition $SQ = Q\sigma$ holds for Π_2 over the entire plane. The quasi-interpolant operator Q over the triangle/quad grid is given by:

$$Qf = \begin{cases} Q^- f = f - \frac{1}{6}(f_{xx} + f_{yy}) & x \leq 0 \\ Q^+ f = f & x > 0 \end{cases}$$

where Q^- and Q^+ satisfy the required equation (3), with $m = 2$, for the left and right schemes, respectively. Given this choice of Q, the special subdivision rules near the y-axis are defined by requiring

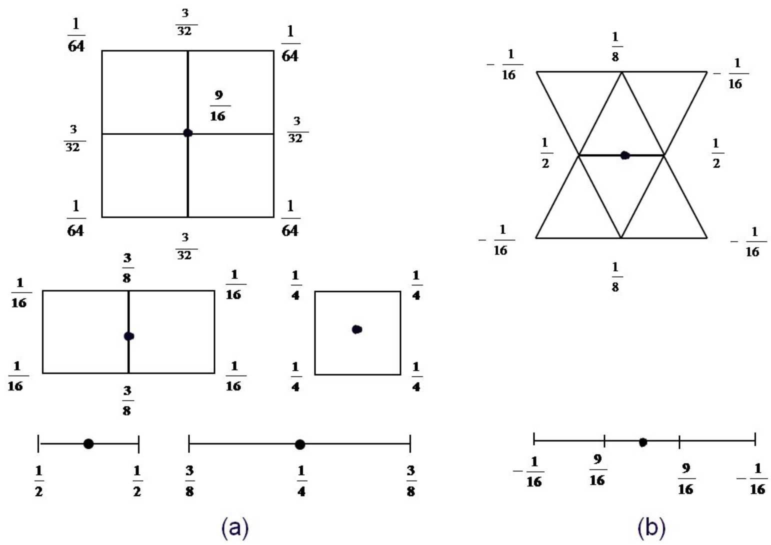

Figure 3: *The stencils in the regular case:(a) the Catmull-Clark scheme, (b) the Butterfly scheme.*

the conditions (10), for m = 2. The choice $Q = Q^-$ on the y-axis is somewhat arbitrary. Different choices of Q lead to different subdivision rules. By experimenting with other choices of Q on the y-axis, we found that for some of them there does not exist subdivision schemes S. So for the choice $Q = Q^-$ on the y-axis the system of equations coming out of (5) is solvable, but the solution is not unique. The challenge is to find a scheme with a support as small as possible which fulfills (10)(shown in Fig. 5.).

3.1 Construction

To construct the subdivision operators S, over the supports $\Omega = \{(i,j), i < 0, j \in \mathbb{Z}\} \cup \{(i,\tilde{j}), i \geq 0, j \in \mathbb{Z}\}$

with: $\tilde{j} = \begin{cases} j + \frac{1}{2}i, & i \leq 0 \\ j, & i > 0 \end{cases}$

we apply the Catmull-Clark subdivision scheme for $i \leq 0$, and the Butterfly subdivision scheme for $i > 0$. Therefore, we need to redefine S for all $(i,j) \in X$ such that $-1 \leq i \leq 2$ (shown Fig. 4.).

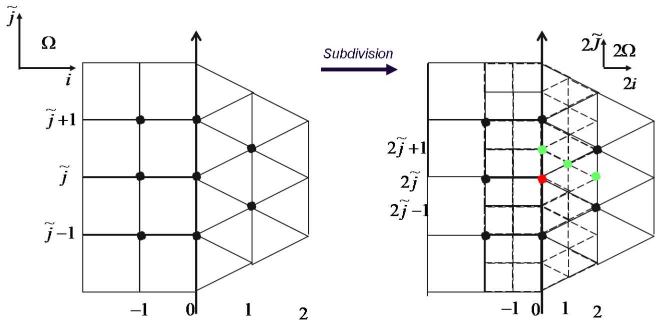

Figure 4: *the tri/quad grid in the left and it's refined grid in the right.*

Construction of $(SP)_{(i,j)}$ near the y-axis

Consider a portion of a regular triangular net with vertices $P_{(i,\tilde{j})}, i \in [0,2], \tilde{j} = \begin{cases} j + \frac{1}{2}i, & i \leq 0 \\ j, & i > 0 \end{cases}, j \in \mathbb{Z}$. The points $(SP)_{(i,\tilde{j})}$ of the refined net can be classified into two distinct groups.

- The edge-points $(SP)_{(1,2\tilde{j}-1)}$ (Fig. 5-(c)):

$$(SP)_{(1,2\tilde{j}-1)} = a_{00}P_{(0,\tilde{j}+1)} + a_{01}P_{(0,\tilde{j})} + a_{02}P_{(0,\tilde{j}-1)} + a_{03}P_{(1,\tilde{j})} + a_{04}P_{(1,\tilde{j}-1)} + a_{05}P_{(2,\tilde{j}-1)}. \quad (11)$$

- The edge-points $(SP)_{(1,2\tilde{j})}$(Fig. 5-(d))

$$(SP)_{(1,2\tilde{j})} = a_{11}P_{(0,\tilde{j}+1)} + a_{12}P_{(0,\tilde{j})} + a_{13}P_{(0,\tilde{j}-1)} + a_{14}P_{(1,\tilde{j}+1)} + a_{15}P_{(1,\tilde{j})} + a_{16}P_{(1,\tilde{j}-1)} + a_{17}P_{(2,\tilde{j}-1)}. \quad (12)$$

Construction of $(SP)_{(i,j)}$ on the y-axis

Consider a portion of a regular quad/triangle net (valence=5) with vertices $P_{(i,\tilde{j})}, i \in [-1,1], \tilde{j} = \begin{cases} j & i \leq 0 \\ j + \frac{1}{2}i & i > 0 \end{cases}, j \in \mathbb{Z},$. The points $(SP)_{i,\tilde{j}}$ of the refined net can be classified into two distinct groups:

- The vertex points $(SP)_{(0,2\tilde{j})}$ (Fig. 5-(a)):

$$(SP)_{(0,2\tilde{j})} = a_{21}P_{(-1,\tilde{j}+1)} + a_{22}P_{(-1,\tilde{j})} + a_{23}P_{(-1,\tilde{j}-1)} + a_{24}P_{(0,\tilde{j}+1)} + a_{25}P_{(0,\tilde{j})} + a_{26}P_{(0,\tilde{j}-1)} + a_{27}P_{(1,\tilde{j})} + a_{28}P_{(1,\tilde{j}-1)}. \quad (13)$$

• The edge-points $(SP)_{(0,2\tilde{j}+1)}$(Fig. 5-(b))
$(SP)_{(0,2\tilde{j}+1)} = a_{31}P_{(-1,\tilde{j}+1)} + a_{32}P_{(-1,\tilde{j})} + a_{33}P_{(0,\tilde{j}+1)}$

$$+a_{34}P_{(0,\tilde{j})} + a_{35}P_{(1,\tilde{j}+1)} + a_{56}P_{(1,\tilde{j})} + a_{57}P_{(1,\tilde{j}-1)}. \qquad (14)$$

Fig. 5 depicts the new stencils defined overs the smallest supports on the y-axis and near it. Due to the invariance under integer shifts in the upwards direction, we only need to compute four new stencils (two stencils near the y-axis and two stencils on the y-axis (i=0)).

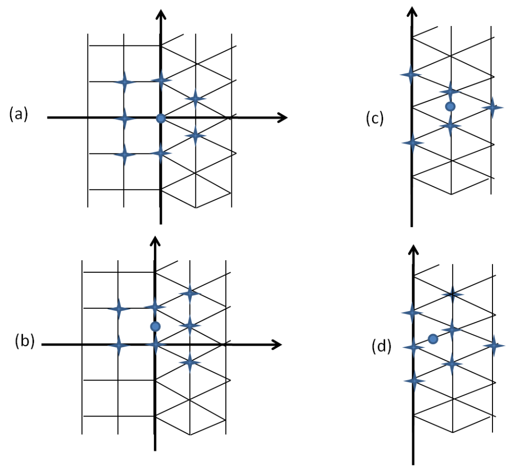

Figure 5: *The weight stencils in the quad/triangle boundary with ordinary support. Control points marked by circles need to be joined to the support such that equations (11),(12),(13) and (14)*

Example

In this example, we show how to find the weight stencil a_{ij} given in equation (13).

We define a weight stencil $w_{(i,j)} \in l(\mathbb{Z}^s)$ at $(0,0)$. To solve for this single weight stencil, we fix a finite support

$\Omega = \{(-1,1),(-1,0),(-1,-1),(0,1),(0,0),(0,-1),(1,\frac{1}{2}),(1,-\frac{1}{2})\}$

defined over the grid X (shown in Fig. 5-(a).), where $w_{(0,0)}$ is assumed to be zero outside of the support Ω.

(13) $\Leftrightarrow SP_{0,0} = a_{21}P_{-1,1} + a_{22}P_{-1,0} + a_{23}P_{-1,-1} + a_{24}P_{0,1}$

$$+a_{25}P_{0,0} + a_{26}P_{0,-1} + a_{27}P_{1,\frac{1}{2}} + a_{28}P_{1,-\frac{1}{2}} = wP_{i,j}, \qquad (15)$$

using equation (10), the solution of equation (15) is given by:

$$SQf(0,0) = \frac{1}{2^{i+j}} Qf(0,0), \forall f(x,y) = x^i y^j \in \Pi_2$$

$$\Rightarrow wQf(x,y)_{/\Omega} = \frac{1}{2^{i+j}} Qf(0,0), \tag{16}$$

we replace Q by $Qf = \begin{cases} Q^- f = f - \frac{1}{6}(f_{xx} + f_{yy}) & x \leq 0 \\ Q^+ f = f & x > 0 \end{cases}$,

and f by $1, x, y, xy, x^2, y^2$.

It is easy to show that the solution of equation (16) is given by: $M_{Q(\Omega)} w_{(0,0)} = d_{Q(0,0)}$. Where $M_{Q(\Omega)}$ denotes a $C^2_{2+2}|\Omega| = 6 \times 8$ matrix and d a $C^2_{2+2} = 6$-column vector with entries

$$M_{ij} = Q(x^i y^j)(\Omega) = \left(\begin{array}{cccccc|cc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 1 & 0 & -1 & \frac{1}{2} & -\frac{1}{2} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 & 1 \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{12} & -\frac{1}{12} \\ -1 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array} \right)$$

$$d_{i+j} = 2^{i+j} Q(x^i y^j)(0,0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -\frac{1}{12} \\ -\frac{1}{12} \\ 0 \end{pmatrix}, \text{ and}$$

$w = (a_{21}, a_{22}, a_{23}, a_{24}, a_{25}, a_{26}, a_{27}, a_{28})$

We can show that given this system of equations, there exists an infinite number of solutions. For symmetry considerations, we therefore suppose that $a_{21} = a_{23}, a_{22} = a_{24} = a_{26}, a_{27} = a_{28}$, then the solution of the weight stencil given by equation (13)is given by: $w = (\frac{3}{224}, \frac{9}{112}, \frac{3}{224}, \frac{9}{112}, \frac{5}{8}, \frac{9}{112}, \frac{3}{56}, \frac{3}{56})$.

Using the same method for equations (11), (12) and (14), we have a scheme with a small support, though probably not the smallest possible, described by the rules shown in Fig. 6.

3.2 Boundaries

When we encounter boundary vertices, we need to use boundary stencils that are usually different from the interior stencils. It is

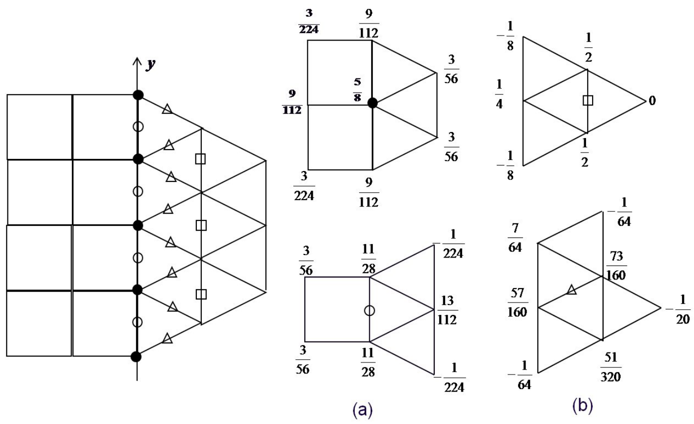

Figure 6: Templates of approximating/interpolatory schemes:(a) the stencils on the y-axis, (b) the stencils near the y-axis.

important that subdivision at any point on the boundary be independent of any point in the interior of the mesh. This permits two surfaces to be joined along a boundary curve. Therefore, cubic B-spline subdivision stencils for curves can be used as the boundary stencils of the Catmull-Clark subdivision scheme and the four point subdivision stencils can be used as the boundary stencils of Butterfly scheme.

In this section, we construct a non-uniform univariate scheme which coincides with the cubic B-spline scheme on the left-half plane, and the four point subdivision scheme on the right half-plane and we gives the rules extending our subdivision scheme to meshes with boundary. From theorem (1) we know that it is sufficient to show that for some $Q : \Pi_3(\mathbb{R}) \to l(\mathbb{Z})$,

$$SQ = Q\sigma.$$

For a given set of control point (P) we define a set of new control point $(SP)_x$ over the support $[-\alpha, \alpha], \alpha \in \mathbb{R}^+$ for $x \leq 0$ by the cubic B-spline scheme, and for $x > 0$ by the four point scheme. The quasi-interpolant operator Q over the support $[-\alpha, \alpha]$ is given by:

$$Qf(x) = \begin{cases} f(x) - \frac{1}{6} f_{xx} & x \leq 0 \\ f(x) & x > 0 \end{cases} \quad (17)$$

Given this choice of Q, the special subdivision rules on the origin point and near it, are defined by requiring the conditions (4), for $m = 3$.

So the non-uniform univariate scheme over the support $[-\alpha, \alpha]$ is described by the rules shown in Fig.6.

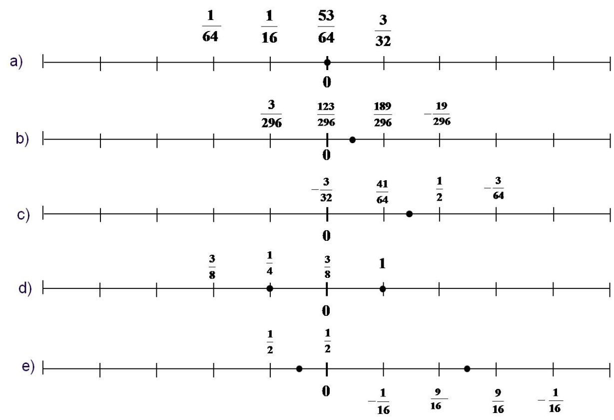

Figure 7: *A non-uniform subdivision scheme combining the four point scheme with the cubic B-spline scheme. (a, b, c) the creases in the common boundary. (d, e) the stencils of B-spline cubic on the left and the stencils of the four point in the right.*

Note that the rules extending our subdivision scheme to meshes with boundary are somewhat more complex, because the support of our subdivision scheme and the butterfly scheme are larger (for $x \geq -1$). A number of different cases have to be considered separately. A complete set of rules for a mesh in the boundary is given in Figures 7, 9.

4 Eigen Analysis of Subdivision

4.1 Necessary Conditions

Of crucial importance in the theory of subdivision surfaces is the subdivision matrix S. The eigen structure of this matrix is im-

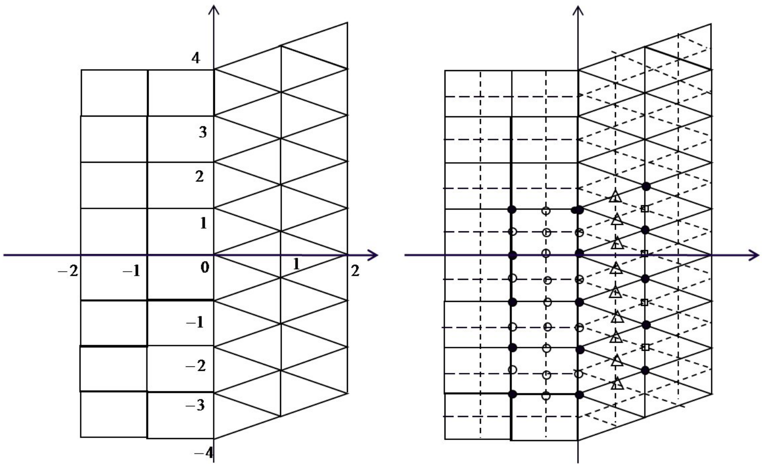

Figure 8: *Quad/triangle mesh (left) and refined quad/triangle mesh (right) .*

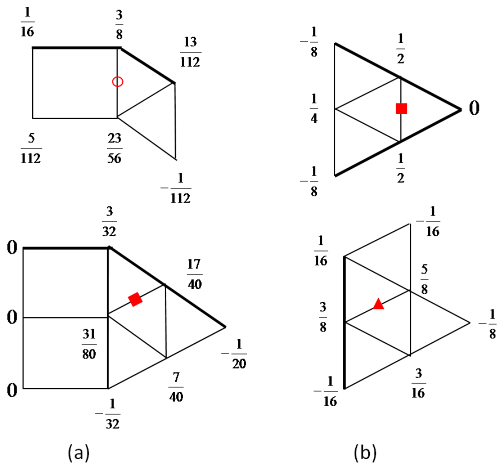

Figure 9: *The creases of our subdivision scheme in the boundary: (a) the creases in the quad/triangle boundary, (b) the creases of Butterfly.*

portant in the analysis of the limit behavior of the surface at the central vertex. Due to the property of affine invariance, the matrix S always has a maximum eigenvalue λ_0 equal to one. The next five eigenvalues in order of magnitude:

$$\lambda_0 = 1 > \lambda_1 \geq \lambda_2 > \lambda_3 \geq \lambda_4 \geq \lambda_5 > |\lambda_i|, i > 5,$$

are important in characterizing the behavior of the tangent plane and the behavior of the curvature at the central vertex. In particular, the two left eigenvectors corresponding to λ_1 and λ_2 can be used to compute the normal of the limit surface at the central vertex. When the surface curvature is continuous, $\lambda_3 = \lambda_1^2$.

We define S to be the subdivision matrix for approximating interpolatory scheme. For the subdivision scheme to be C^2 in the functional sense, S must satisfy

$$SZ_i = \lambda_i Z_i \tag{18}$$

where $\lambda_i = 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$ and Z_i are the corresponding eigenvectors producing the polynomials $1, x, y, xy, x^2, y^2$. The others eigenvalues $\lambda_i, i > 5$ must be strictly smaller than $\frac{1}{4}$.

4.2 Sufficient Conditions

To analyze the smoothness of our subdivision scheme, we use the joint spectral radius test described by Levin/Levin [5] and used thereafter by Schaefer-Warren [8], Hakenberg [3] and Jiang et al [1] to prove the sufficient conditions of C^2 continuity. This smoothness test requires that the subdivision scheme is C^2 away from the boundary edge and that the subdivision matrix for a point on the boundary satisfies the necessary conditions from section 4.1. Furthermore, the subdivision scheme along the edge must satisfy a joint spectral radius condition. To perform the joint spectral radius test, we require two subdivision matrices (A and B) that map an edge M on the boundary to two smaller edges (M_1 and M_2) after one round of subdivision. The matrices A and B should contain all of the vertices that influence the surface over the edges M_1 and M_2. For that we define the set L which is the set of $|L| = 45$ points (show in Fig. 7.) such that:

$L = \{(i, j) : i = -1, -2, -4 \leq j \leq 4, j \in Z\} \cup \{(i, j + \frac{1}{2}i) : 0 \leq i \leq 2, -4 \leq j \leq 4, j \in Z\}$

The set L denote a subset of mesh points around the origin. Let A denote the subdivision matrix with values from L to L after one subdivision iteration, and let B denote the subdivision matrix with values from L to ξL, where ξ is a shift operator, $\xi L = \{(i, j + 1) \setminus (i, j) \in L\}$. Next, we find a diagonalizing matrix V such that

$$
\begin{aligned}
V^{-1}AV &= \begin{pmatrix} \Lambda & C_0 \\ 0 & Y_0 \end{pmatrix} \\
V^{-1}BV &= \begin{pmatrix} \Theta & C_1 \\ 0 & Y_1 \end{pmatrix}
\end{aligned} \qquad (19)
$$

where $\Lambda = diag(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and Θ is an upper-triangular matrix with the same diagonal entries as Λ. The upper-triangular matrix for our scheme is:

$$
\Theta = \begin{pmatrix}
1 & \frac{4}{25} & \frac{19}{100} & -\frac{3}{50} & \frac{1}{20} & -\frac{1}{20} \\
0 & \frac{1}{2} & 0 & -\frac{19}{100} & \frac{9}{100} & -\frac{19}{100} \\
0 & 0 & \frac{1}{2} & -\frac{4}{25} & \frac{4}{25} & -\frac{13}{100} \\
0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{4}
\end{pmatrix}.
$$

The obvious choice for constructing the matrix V is to simply use all of the eigenvectors of A. However, this approach can be numerically unstable if the matrix has small eigenvalues. The matrix V is formed from the right eigenvectors associated with the eigenvalues from Λ and a basis of the null space of the corresponding left eigenvectors.

Finally, we use Y_0 and Y_1 to compute $\rho^{[k]}(Y_0, Y_1) = (Max\|Y_{\epsilon_k} Y_{\epsilon_{k-1}} ... Y_{\epsilon_1}\|_\infty)^{\frac{1}{k}}$ where $\epsilon_i \in \{0, 1\}$.

From [5], if there exists k such that $\rho^{[k]} < \frac{1}{4}$, then the quad/triangle subdivision scheme is C^2 at the boundary. Applying the joint spectral radius technique to our quad/triangle subdivision scheme, we find $\rho^{[17]} = \frac{1750}{7017} < \frac{1}{4}$ and our quad/triangle subdivision scheme satisfies the necessary conditions for polynomial generation, so we conclude that our approximating/interpolatory quad/triangle subdivision scheme is C^2 at quad/triangle boundaries.

5 Implementation and Results

Our subdivision scheme is stationary and easy to implement as every classical scheme. We use a large number of stencils at the boundary but their support is compact. In areas where the mesh is quadrilateral and for $x < 0$, we apply the Catmull-Clark algorithm. In areas where the mesh is triangular and for $x > 0$, we apply the butterfly algorithm. For $-1 < x < 2$, along the quad/triangle boundary where vertices in the $y-axis$ are shared by two adjacent quads and three adjacent triangles we apply our stencils given in Fig. 5. In the boundary for the creases we apply the stencils given in Figures 6, 8.

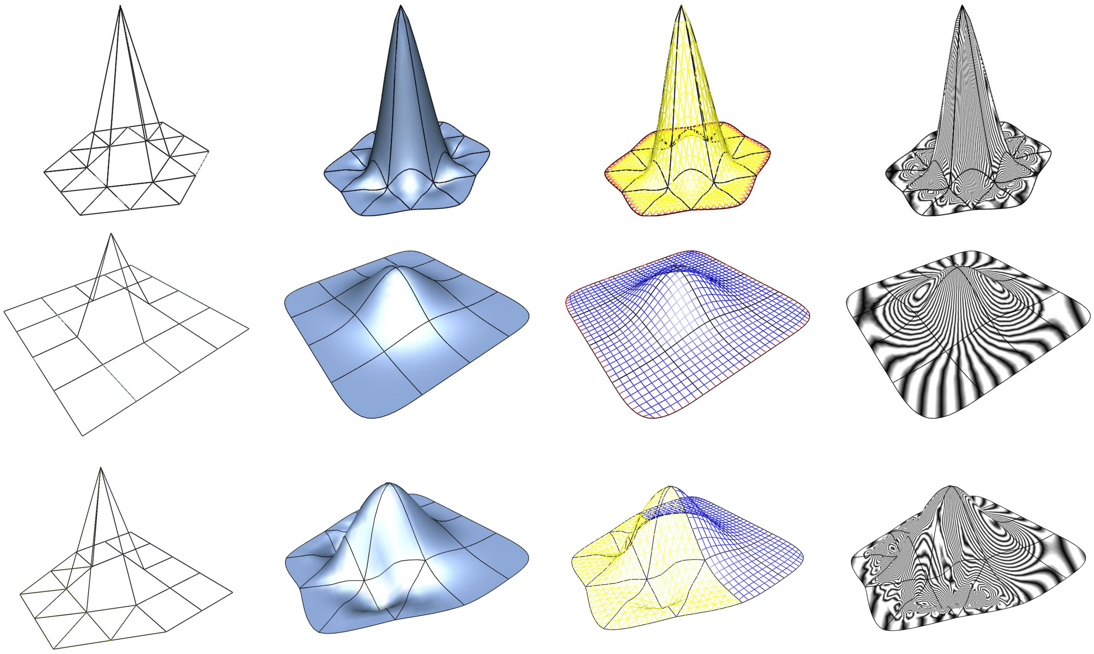

Figure 10: *Comparison of the Butterfly scheme (top), Catmull-Clark scheme (middle) and our new scheme (bottom). From left to right: the control meshes, the limit surfaces, colored meshes (yellow for triangles, blue for quads), and reflexion lines.*

Figure 9 demonstrates that our new scheme performs as well as the Catmull-Clark and Butterfly schemes on a pic-like polygonal model.

Figure 10 depicts different surfaces created using our subdivision scheme. Note that the right-most pictures show a reflection-line plot of the surface, which provides an excellent quality of the surface generated by our subdivision scheme along the quad/triangle boundary.

6 Conclusion

In this paper we have proposed a new quad/triangle subdivision scheme which unifies the Catmull-Clark and the Butterfly subdivision algorithms along the quad/triangle boundary. On the boundary vertices are shared by two adjacent quads and three adjacent triangles. We have proven that our scheme is C^2 everywhere except for the ordinary Butterfly where our scheme is C^1. Our method is based on the theory of polynomial generation. In areas where the mesh is regular, the subdivision operator S^∞ generates polynomials of degree up to m, and the equation of limit surface is known, thus the continuity can deduced. On the contrary in areas where the mesh is irregular , we can not find the exact equation generated by S^∞. We must therefore propose rules which optimize the behavior of the curvature at the boundary in the irregular case. In the future we intend to find a general mask (any valence) and to give a formal proof of continuity. This could be performed by introducing a general mask followed by a correction mask or by a novel method based on properties of polynomial generation near a singular point.

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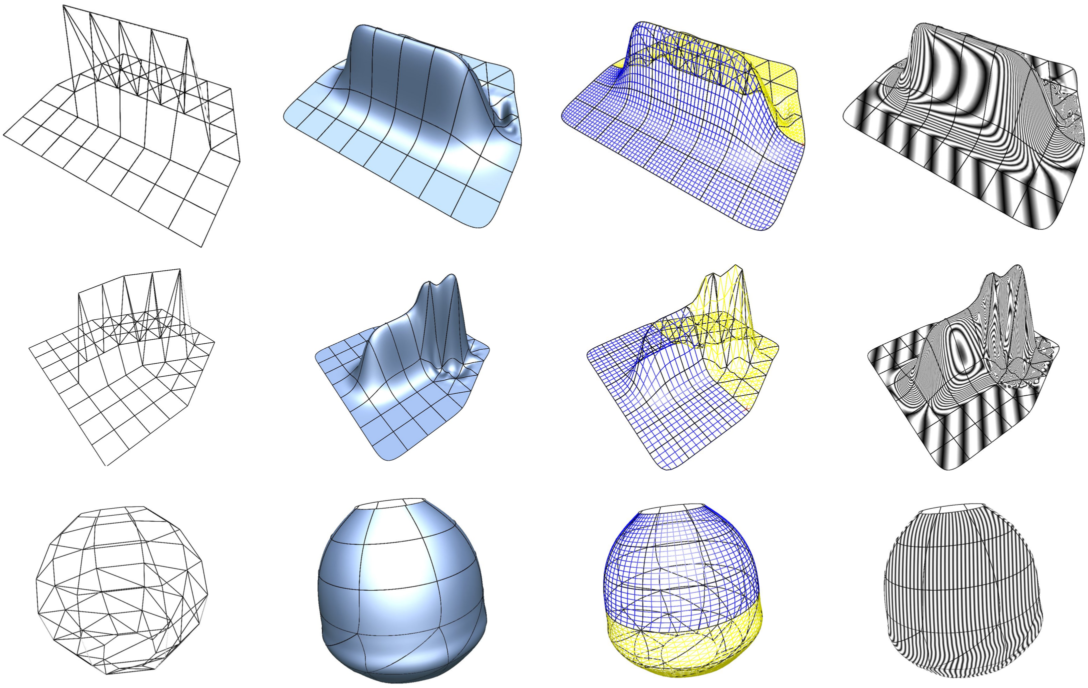

Figure 11: *Different surfaces generated using our new scheme.*

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