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Gluing Dupin cyclides along circles, finding a cyclide given three contact conditions.

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Abstract

Dupin cyclides form a 9-dimensional set of surfaces which are, from the viewpoint of differential geometry, the simplest after planes and spheres. We prove here that, given three oriented contact conditions, there is in general no Dupin cyclide satisfying them, but if the contact conditions belongs to a codimension one subset, then there is a one-parameter family of solutions, which are all tangent along a curve determined by the three contact conditions.

1 Introduction

Curved patches have been an important tool in design since the introduction of digital computers in engineering. Among them triangular Bézier patches are a natural extension of Bézier curves and have a very elegant mathematical presentation in terms of barycentric coordinates. See Farin´s essay: A History of Curves and Surfaces in CAGD in the Handbook of Computer Aided Geometric Design [Fa-Ho-Myu] for a short but informative historical sketch of triangular patches and also chapter 17 of his classical book [Fa], for a short introduction to the subject and the references therein. Triangular patches are also useful do reduce the size of a triangulation by replacing sets of planar triangles with approximating curved triangles. Frequently it is important that a triangular patch interpolates three given points and has prescribed tangent planes at these points. It is also desirable that patches have the lowest degree possible and are rationally parameterizable, ideally to be triangular pieces of quadrics. For arbitrary contact conditions this in general is not possible as observed in [B-H]. The next best thing are patches of Dupin cyclides which have parametric and algebraic degree less than or equal to four ([1]). Four sided Dupin cyclide patches have been introduced by R.R. Martin [Ma1, Ma2] and more recently [D-G-L-M-B] used rings of Dupin cyclides to join spheres, planes and canal surfaces.

The result of the present article is an essential preliminary step before being able to match G^1 patches of Dupin cyclides along curves which are not characteristic circles.

2 Generalities about Dupin cyclides

Rotating a circle contained in $\mathbb{R}^3$ around and axis contained in the plane of the circles we may obtain three types of surfaces (see fig 1)

- a ring torus when the axis does not intersect the circle
- a surface with one singular point when the axis is tangent to the circle (horn torus)
- a surface with two conical singular points when the axis intersect the circle in two points (spindle torus).

To obtain a larger, but still geometrically interesting family of surfaces, we consider the image of the tori of revolution under the action of an element of the Möbius group $g \in \mathcal{M}$. For us the Möbius group acts on $\mathbb{R}^3$; wherever the action of $g \in \mathcal{M}$ is defined, its differential is a similarity. Figure 2 show a Dupin cyclide with two singular points from the inside of a compact component of its complement in $\mathbb{R}^3$.

From the viewpoint of differential geometry, Dupin cyclides are somehow the simplest surfaces after the planes, spheres, as, on a Dupin cyclide, the two principal curvatures k_1 and k_2 are constant along the corresponding lines of curvature (quadrics, from the algebraic viewpoint, may also be considered as surfaces simpler that Dupin cyclides, although their curvature functions are complicated).

[1] Algebraic degree three patches have been studied in [Mu]

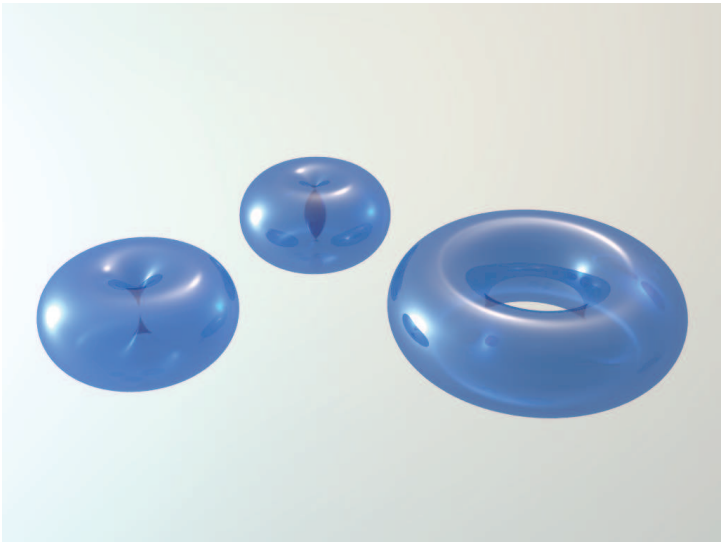


Figure 1: The three types of tori of revolution

3 Dupin cyclides as envelopes of spheres

Let us now present a directly conformal definition of the Dupin cyclides as envelopes of very particular one-parameter family of spheres.

3.1 The space of oriented spheres

The *Lorentz quadratic form* $\mathcal{L}$ on $\mathbb{R}^5$ and the associated *Lorentz bilinear form* $\mathcal{L}(\cdot,\cdot)$, are defined by $\mathcal{L}(x_0,\cdots,x_4) = -x_0^2 + (x_1^2 + \cdots + x_4^2)$ and $\mathcal{L}(u,v) = -u_0v_0 + (u_1v_1 + \cdots + u_4v_4)$.

The Euclidean space $\mathbb{R}^5$ equipped with this pseudo-inner product $\mathcal{L}$ is called the *Lorentz space* and denoted by $\mathbb{R}^5_1$.

The isotropy cone $\mathcal{L}ight = \{v \in \mathbb{R}^5_1 \,|\, \mathcal{L}(v) = 0\}$ of $\mathcal{L}$ is called the *light cone*. Its non-zero vectors are also called *light-like* vectors. The *light-cone* divides the set of vectors $v \in \mathbb{R}^5_1, v \notin \mathcal{L}ight$ in two classes:

A vector v in $\mathbb{R}^5_1$ is called *space-like* if $\mathcal{L}(v) > 0$ and *time-like* if $\mathcal{L}(v) < 0$.

A straight line containing the origin is called space-like (or time-like) if it contains a space-like (or respectively, time-like) vector. A subspace through the origin of dimension k is called space-like if its contains only space-like vectors, time-like if it contains one time-like vector; it will then contain a $(k-1)$-cone of light-like vectors. It is called light-like if it is tangent along a generatrix to the light-cone; in that case, the generatrix direction is the unique light-like direction of the subspace. We will use the same terminology when affine subspaces are involved.

To get a model $\mathbb{E}^3$ of the Euclidean space $\mathbb{R}^3$, we need to slice the light cone by an affine light-like hyperplane (see Figure 3 and, again, [H-J] and [L-W]). Such an hyperplane intersects all the rays of the cone but one. We refer to this ray as corresponding to the *point at infinity* of $\mathbb{E}^3$ (we get a 3-dimensional sphere adding a point "∞" to $\mathbb{E}^3$).

To each point $\sigma \in \Lambda^4 = \{v \in \mathbb{R}^5_1 \,|\, \mathcal{L}(v) = 1\}$ corresponds an oriented sphere or plane $\Sigma = \sigma^\perp \cap \mathbb{E}^3$ (see Figure 4). The orientation of Σ determines a component B of $\mathbb{E}^3 \setminus \Sigma$ such that Σ is oriented as the boundary of B.

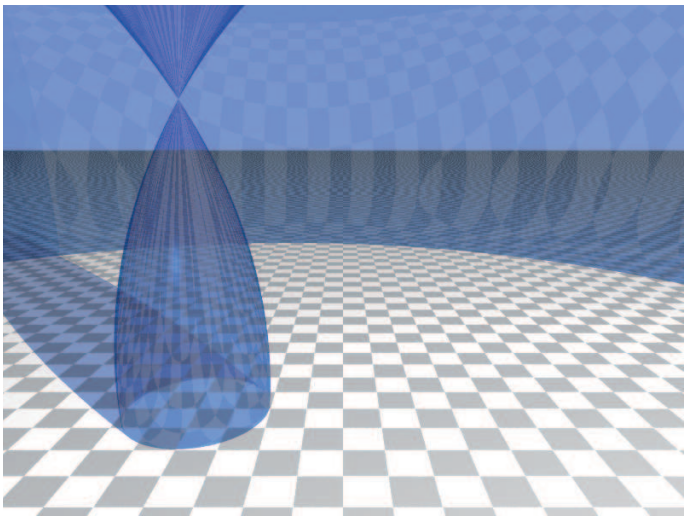


Figure 2: View from the interior of a singular locus

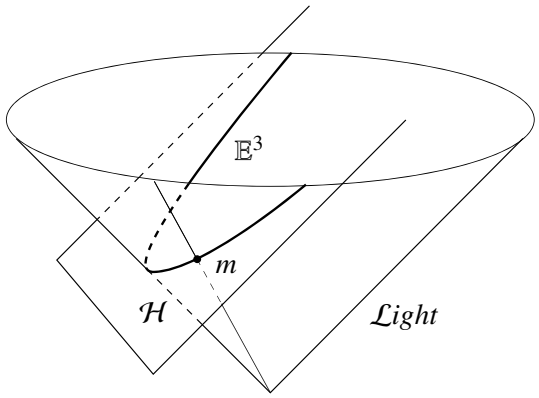


Figure 3: Euclidean models in the Lorentz space $\mathbb{R}^5_1$.

We can see Λ^4 as a sphere for the Lorentz quadratic form; as for the usual Euclidean sphere, the tangent hyperplane $T_\sigma\Lambda^4$ is orthogonal to σ (seen as a vector of $\mathbb{R}^5_1$). It is time-like and $T_\sigma\Lambda^4 \cap \Lambda^4$ is a cone of dimension 3 formed of (affine) light-rays. Again tangent vectors at a point $\sigma \in \Lambda^4$ are of one of the three types: space, light or time.

Consider the hyperplanes $T_{\sigma_1}\Lambda^4$ tangent to Λ^4 at σ_1 and $T_{-\sigma_1}\Lambda^4$. The intersections $\Lambda_4 \cap T_{\pm\sigma_1}\Lambda_4$ are cones of dimension 3. Depending to the location of the sphere σ_2 according to these cones, we can see if its intersects or not the first sphere Σ_1, see Figure 5:

- if σ_2 belongs to the "exterior" of both cones Σ_2 intersects Σ_1 in a circle; in this case $|\mathcal{L}(\sigma_1, \sigma_2)| < 1$.
- if σ_2 belongs to one of the cones, the two spheres are tangent; then $|\mathcal{L}(\sigma_1, \sigma_2)| = 1$.
- if σ_2 belongs to the "interior" of one of the cones the two spheres are disjoint; then $|\mathcal{L}(\sigma_1, \sigma_2)| > 1$.

The position of σ in one of the four different parts of the "interiors" of the two cones distinguish different cases of intersection of the balls $B_1; \Sigma_1 = \partial B_1$ and $B_2; \Sigma_2 = \partial B_2$. If σ_2 is in the "interior" of the cone C_{σ_1} of vertex σ_1, the balls B_1 and B_2 intersect. If σ_2 belongs to the bottom part of the "interior" of

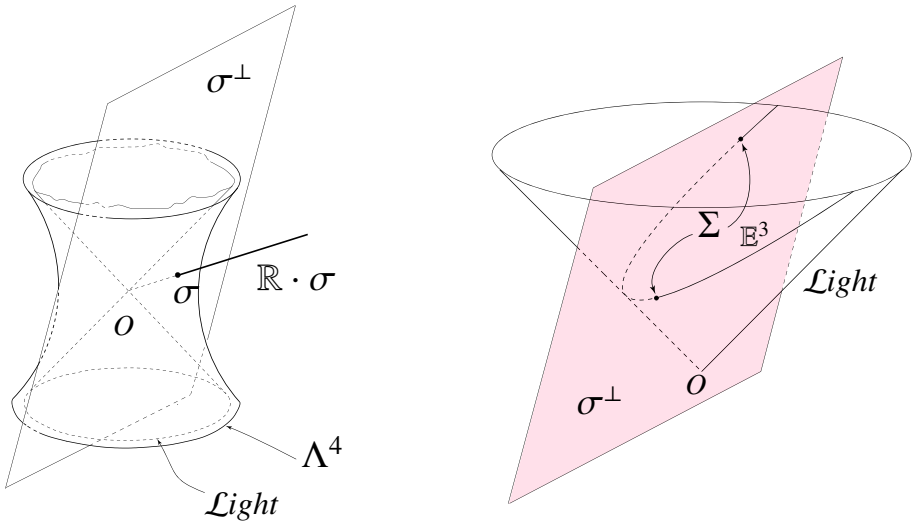


Figure 4: The correspondence between Λ^4 and the space of spheres or planes in $\mathbb{E}^3$.

C_{σ_1}, then $B_1 \cap B_2 = B_1$. If it belongs to the upper part, $B_1 \cap B_2 = B_2$. If σ_2 is in the "interior" of the cone $C_{-\sigma_1}$ of vertex $-\sigma_1$, the balls B_1 and B_2 are disjoint when σ_2 belong to the upper side of the "interior" of $C_{-\sigma_1}$. When σ_2 belongs to the lower component, the intersection $B_1 \cap B_2$ is a zone $B_1 \cap B_2 \simeq \mathbb{S}^2 \times [0,1]$ (see Figure 5).

Notice that the restriction to a space-like vectorial plane P_{space} containing the origin is, for the metric induced from the ambient Lorentz quadratic form, a positive definite quadratic form. We consider it as the Euclidean structure of P. The intersection of Λ^4 with P_{space} is, for the metric induced from $\mathcal{L}$ on P_{space}, a circle $\gamma \subset \Lambda^4$ of radius one. The center of this circle is the origin.

The points of this circle correspond to the spheres of a pencil with base circle. The arc-length of a segment contained in γ is equal to the angle between the spheres corresponding to the extremities of the arc. This fact can be verified using the formula $\sigma = km + n$, where k is the curvature of the sphere Σ corresponding to the point $\sigma \in \Lambda^4$ (k=0 if Σ is a plane), $m \in \mathbb{E}^3 \subset \mathcal{L}ight$ is a light-like vector, and n is the unit vector contained in $T_m\mathbb{E}^3 \simeq \mathbb{R}^3$ normal to Σ (see [H-J] or [L-O] p. 276–278). As m is orthogonal to $n \in T_m\mathbb{E}^3 \subset T_m\mathcal{L}ight$, we see that $\mathcal{L}(\sigma_1, \sigma_2) = (n_1 \,|\, n_2) = cos\theta$, where θ is the angle between n_1 and n_2, that is the angle of Σ_1 and Σ_2.

The intersection of Λ^4 with a time-like vectorial plane P_{time} is not connected. For an Euclidean eye, seeing as orthonormal basis a Lorentz "unitary" basis, it is an equilateral hyperbola. We will often still refer to such a hyperbola as a "circle". The origin is its center both for an Euclidean eye and with the Lorentz viewpoint: the Lorentz distance from any point of the hyperbola to the origin is equal to 1. The points of this "circle" correspond to spheres of a Poncelet pencil or pencil with distinct limit point. The two light-directions of P_{time}correspond to the two limit points of the pencil.

The intersection of Λ^4 with a light-like plane P_{light} is the union of two parallel affine light-rays. The corresponding spheres form a pencil of spheres tangent to the point corresponding to the direction of the light-rays.

3.2 Intersection of Λ^4 with affine planes

In this subsection, P is an affine plane of $\mathbb{R}^5_1$.

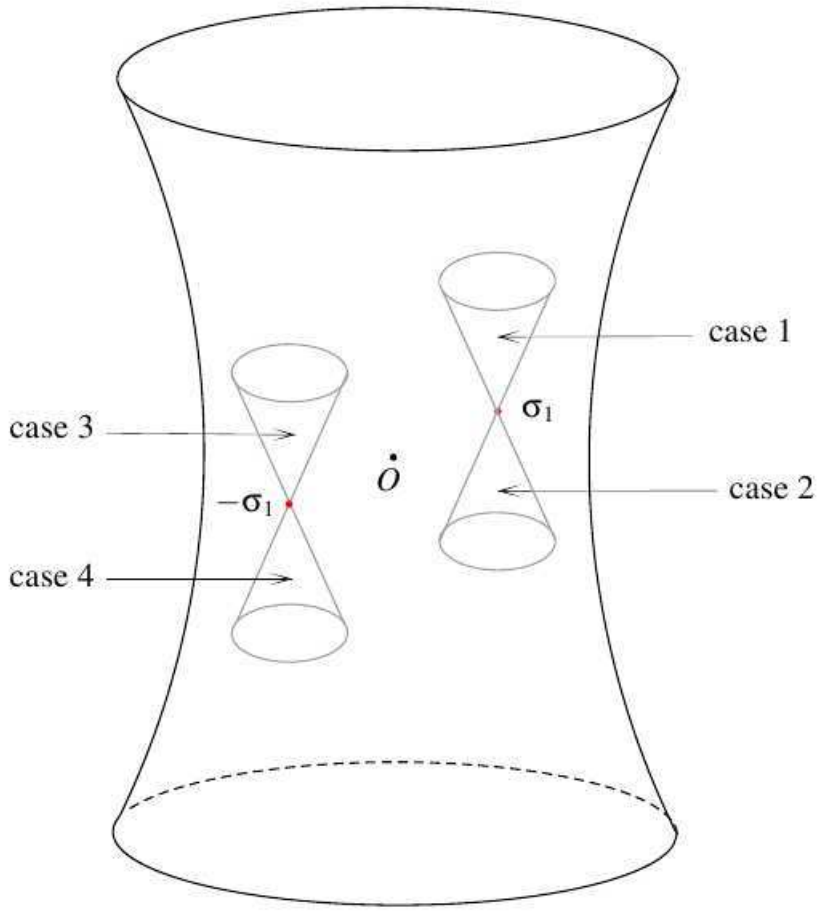


Figure 5: The two cones C_{σ_1} and $C_{-\sigma_1}$. Case 1: $B_1 \cap B_2 = B_2$. Case 2: $B_1 \cap B_2 = B_1$. Case 3: $B_1 \cap B_2 = \varnothing$. Case 4: $B_1 \cap B_2 \simeq \mathbb{S}^2 \times [0, 1]$.

Let us first study the intersection $P \cap \Lambda^4$ when P is space-like. The orthogonal projection $\pi(O)$ of the origin O on P achieves the minimum value of $\mathcal{L}(x), x \in P$. This value $\eta = min(\mathcal{L}(x), x \in P)$ maybe negative, zero or positive.

- If $\eta < 0$, the intersection is a circle of radius r larger that one; $r^2 = 1 - \eta$. We will see (subsection 3.4) that the envelope is a regular Dupin cyclide.

- If $\eta = 0$, the circle is of radius 1. Either the plane is a vectorial plane and the spheres corresponding to the points of $\gamma = \Lambda^4 \cap P$ form a pencil, or if P does not contain the origin, the spheres corresponding to the points of $\gamma = \Lambda^4 \cap P$ are tangent to a given direction at a point m. We will see (Subsection 3.4) that, when P does not contain the origin, the envelope is a Dupin cyclide with a unique singular point.

- If $0 < \eta < 1$, the circle is of radius r less than 1, $r^2 = 1 - \eta$. The spheres corresponding to the points of $\gamma = \Lambda^4 \cap P$ contain two points which are singular point of the envelope (see Subsection 3.4).

- If $\eta = 1$, the plane P is tangent to Λ^4, and intersects it at one point σ.

- If $\eta > 1$, the intersection $P \cap \Lambda^4$ is empty.

Let us now suppose that P is time-like. The orthogonal projection $\pi(O)$ of the origin on P is now a critical point of the function $\mathcal{L}(x), x \in P$, but not a minimum. Let us still call η the value $\eta = \mathcal{L}(\pi(O))$. This value is positive, as the vectorial space orthogonal to P is space-like.

- If $\eta = 0$, that is if P is a vectorial time-like plane, the intersection $\gamma = \Lambda^4 \cap P$ is the equilateral hyperbola formed of two time-like curves given by the equation $\mathcal{L}(x, y) = -x^2 + y^2 = 1$, where x and y are coordinates defined using a Lorentz "orthonormal" basis of two vectors of the plane, one space-like and one time-like. The corresponding spheres form a Poncelet pencil.

- If $0 < \eta < 1$, $\gamma = \Lambda^4 \cap P$ is still a hyperbola formed of two time-like curves. The corresponding spheres are nested.

- If $\eta = 1$, the plane P is tangent to Λ^4. The intersection $\gamma = \Lambda^4 \cap P$ is the union of two affine light-rays.

- If $\eta > 1$, $\gamma = \Lambda^4 \cap P$ is a hyperbola formed of two space-like curves. The corresponding family of spheres has an envelope which is a surface with two conical singularities. Going to infinity on a branch of the hyperbola correspond to let the radius of the spheres go to zero; the spheres approach a singular point of the envelope (see Figure 2).

Finally, suppose that P is light-like. The intersection $P \cap \Lambda^4$ is a space-like parabola. The corresponding envelope has a unique singular point. The infimum

$$inf_P \mathcal{L}(x),\ x \in P$$

is well defined, but achieved on an affine light-like line.

3.3 Envelopes of one-parameter family of spheres

A surface is always the envelope of a two-parameter family of spheres. Some particular surfaces: the canal surfaces, are envelopes of one-parameter family. For a C^1 curve $\gamma \subset \Lambda^4$, that is a one-parameter family of spheres, to generate a canal surface, an extra condition is needed: the curve should be *space-like*, that is its tangent vector should be everywhere space-like. With no special assumption, the envelope maybe singular.

To guarantee that the canal surface is locally immersed, an extra condition is needed. As the curve γ is time-like, we can suppose it is parameterized by arc-length, that is $\mathcal{L}(\dot{\gamma}(s)) = 1 \forall s$. the geodesic curvature vector is the orthogonal projection of the acceleration on the tangent space to Λ^4. It satisfies the formula $\vec{k_g}(s) = \ddot{\gamma}(s) + \gamma(s)$.

Figure 6 shows the behavior of the spheres corresponding to a space-like curve with time-like geodesic curvature in Λ^4 and to a time-like curve in Λ^4.

Notice that a canal surface is a union of circles, the *characteristic circles*. They are obtained intersecting two spheres, one corresponding to a point $\sigma \in \gamma$, the second to the vector $\dot{\gamma}(\sigma)$ tangent to γ at σ. We can suppose, as the curve γ is space-like, that the vector $\dot{\gamma}(\sigma)$ is a unit vector, that is a point in Λ^4 which therefore corresponds to a sphere.

3.4 A presentation of Dupin cyclides as "circles" in Λ^4

A Dupin cyclide C is in two different ways the envelope of a one-parameter family of spheres (see [Dar] and Figure 8). This implies that it has two families of characteristic circles (see Figure 7).

This implies also (see [L-W], pp. 161–168) that the two corresponding curves are "circles" of Λ^4, that is intersection of Λ^4 with two affine planes that we call *brothers*.

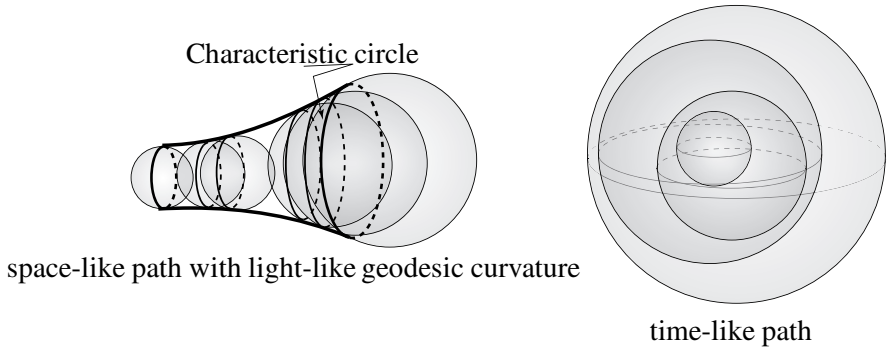


Figure 6: Spheres corresponding to space-like and time-like paths.

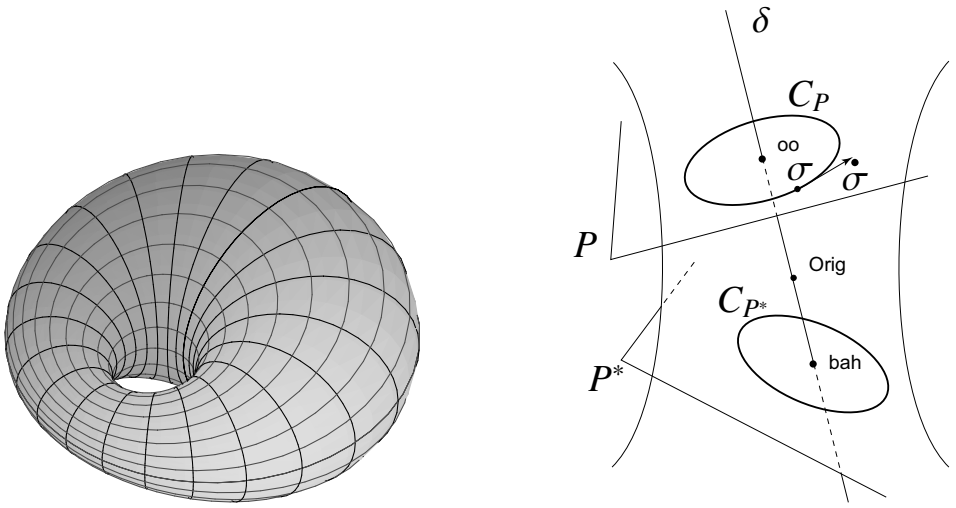


Figure 7: A Dupin cyclide and its two families of characteristic; the two brother "circles" in Λ^4 are circles

Definition 3.4.1. *The brother affine plane P^* of an affine plane P intersection of which with Λ^4 is space-like is defined by*

$$P^* = \{y\} \text{ such that } \forall x \in P, \mathcal{L}(x, y) = -1. \tag{1}$$

Notice that the brother P^{**} of P^* is P.

Remark: The intersection of three four dimensional affine tangent planes $T_{m_i}\Lambda^4$, $i = 1, 2, 3$ at three points $m_i \in P \cap \Lambda^4$ is the 2-dimensional affine plane P^*

Proof: The affine tangent hyperplane $T_\sigma\Lambda^4$ to Λ^4 at a point σ is orthogonal to $\mathbb{R}\cdot\sigma$ at σ. It has therefore the equation $\mathcal{L}(y, \sigma) = 1$. The condition $\forall x \in P, \mathcal{L}(x, y) = 1$ is equivalent to the three conditions $\mathcal{L}(y, \sigma_1) = 1$, $\mathcal{L}(y, \sigma_2) = 1$, $\mathcal{L}(y, \sigma_3) = 1$, for any choice of three different points on the circle $C_P = P \cap \Lambda^4$. □

Equation (1) implies that the brother P^* of P is orthogonal to P. Moreover, the common perpendicular δ contains the origin; we can write $\delta = (p \oplus p^*)^\perp$, where p and p^* are the vectorial planes parallel respectively to P and P^*. In fact, p^* is orthogonal to $span(P)$ and p is orthogonal to $span(P^*)$. The line

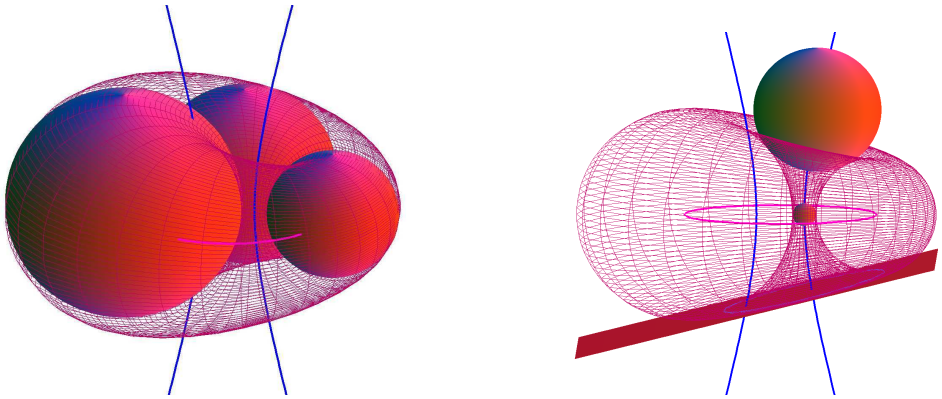


Figure 8: The two familly of spheres having a Dupin cyclide as the envelope

δ is also the intersection $span(P) \cap span(P^*)$. The (vectorial) line δ is time-like when the Dupin cyclide is regular.

Let $C_P = P \cap \Lambda^4$ and $C_{P^*} = P^* \cap \Lambda^4$. The curves C_P and C_{P^*} are the "circles" the points of which correspond to the two families of spheres defining the Dupin cyclide C as an envelope.

Proposition 3.4.2. *The line ℓ_{σ,σ^*} joining a point $\sigma \in C_P$ to a point $\sigma^* \in C_{P^*}$ is a light-ray contained in Λ^4 orthogonal to $\mathbb{C}_P$ at σ and to C_{P^*} at σ^*.*

Proof: The curve C_{P^*} is contained in the cone, of vertex σ, formed of light-rays $T_\sigma\Lambda^4 \cap \Lambda^4$. Therefore a light-ray of this cone joins σ to $\sigma^* \in C_{P^*}$.

Let T be a unit vector tangent to C_P at σ. It is contained in $T_\sigma\Lambda^4$, and therefore is orthogonal to σ. It is contained in p, and therefore is orthogonal to $span(P^*)$, and in particular to σ^*. The vector $\sigma^* - \sigma$ generating the direction of ℓ_{σ,σ^*} is therefore orthogonal to T. □

The direction of the light-ray ℓ_{σ,σ^*} correspond to the point of the Dupin cyclide C where the two spheres corresponding to σ and σ^* are tangent.

When the two brother circles are both Euclidean circles, we can use their angle parameters θ and ψ to get a parameterization of the regular cyclide in $\mathbb{R}^3$ envelope of spheres corresponding to the points of the circles.

$$\Gamma_d(\theta,\psi) = \begin{pmatrix} x(\theta,\psi) \\ y(\theta,\psi) \\ z(\theta,\psi) \end{pmatrix} = \begin{pmatrix} \dfrac{\mu(c - a\cos\theta\,\cos\psi) + b^2\cos\theta}{a - c\cos\theta\,\cos\psi} \\ \dfrac{b\sin\theta\,(a - \mu\cos\psi)}{a - c\cos\theta\,\cos\psi} \\ \dfrac{b\sin\psi\,(c\cos\theta - \mu)}{a - c\cos\theta\,\cos\psi} \end{pmatrix} \tag{2}$$

The characteristic circles are given in the parametric equation (2) by a constant value of θ or ψ, see Figure 7.

The points where the sphere of parameter θ_0 is tangent to the sphere of parameter ψ_0 is the point of intersection of the characteristic circle defined by $\theta = \theta_0$ and $\psi = \psi_0$. The three parameters a, b, c; $c^2 = a^2 - b^2$ are associated to the two *anticonics*[2] set of the centers of the spheres of the two families; their

[2]the two conics are contained in perpendicular planes; moreover the foci of one are the vertices of the other

equations are $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; $\frac{y^2}{c^2} - \frac{z^2}{c^2} = 1$. The parameter μ is related with the axis of the pencils of affine planes which intersect the cyclide along characteristic circles (see [For] and [D-G-L-M-B]).

One can also show that (see for example [L-W] pp. 161–168)

- Any regular Dupin cyclide of $\mathbb{R}^3$ is the image by a Möbius map of a torus of revolution.

Using some complex geometry one can prove a classical result (see [Vi]). Regular Dupin cyclides contain two families of circles which are not characteristic circles: the Villarceau circles. Unlike characteristic circles, where there is only one sphere of the pencil containing the circle which is tangent to the Dupin cyclide (along the whole characteristic circle), all the spheres of the pencil containing a Villarceau circle are tangent to the Dupin cyclide at exactly two points, which depend on the sphere.

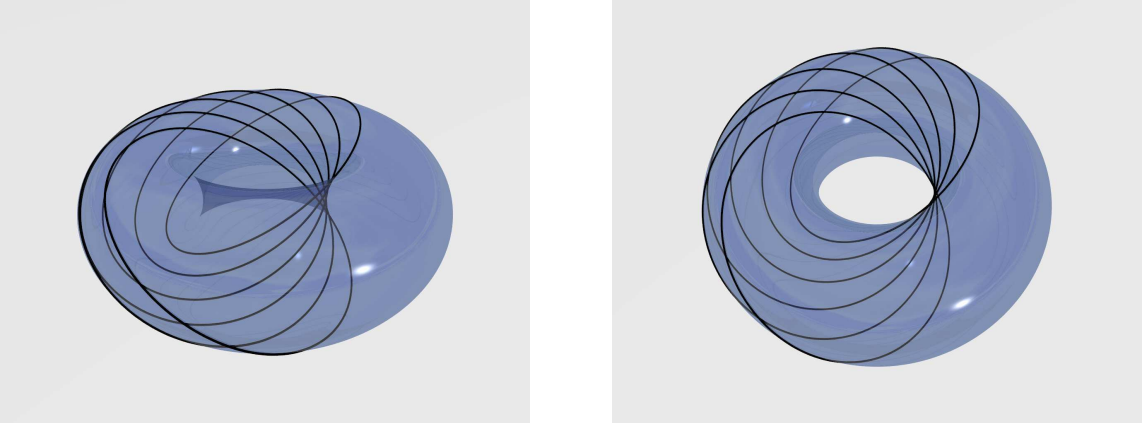


Figure 9: Torus and Villarceau circles

- Any Dupin cyclide with one singular point is conformally equivalent to a revolution cylinder of $\mathbb{R}^3$ completed by the point ∞.

- Any Dupin cyclide with two singular points is conformally equivalent to a cone of revolution of $\mathbb{R}^3$ completed by the point ∞.

Notice finally that Dupin cyclide are algebraic surfaces of degree at most 4.

4 Gluing Dupin cyclides and canal surfaces along a characteristic circle

The previous considerations show that a three-dimensional family of Dupin cyclides are tangent along a given characteristic circle; they correspond to suitable affine planes containing a given pair $\sigma \in \Lambda^4$, $v \in T_\sigma \Lambda^4$.

Gluing C^0 two smooth space-like arcs γ_1 and γ_2 with time-like geodesic curvature (see subsection 3.3) in Λ_4 such that the tangent vectors at the point where the arcs meet make an angle introduces a piece of the sphere corresponding to the angular point of the curve. Depending on how the two characteristic circles corresponding to the two arcs are dispatched on this sphere, the envelope maybe get cuspidal singularities where the annulus of Dupin cyclide meets the sphere. Considering these two characteristic circles, we can have three different relative positions: they can meat each other (Figure 10), they can be tangent (Figure 11), or they can be disjoint (Figure 12). The two canal surfaces and a piece of the sphere cannot be blended G^1 when the two characteristic circles intersect.

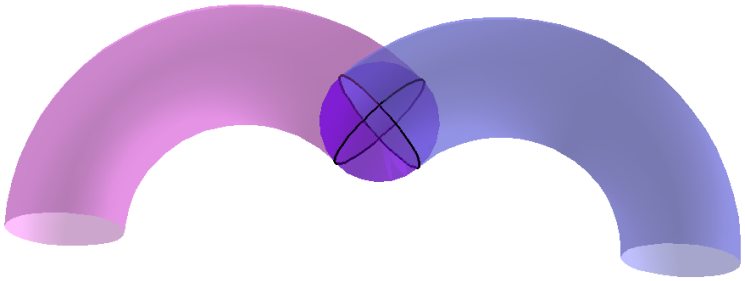
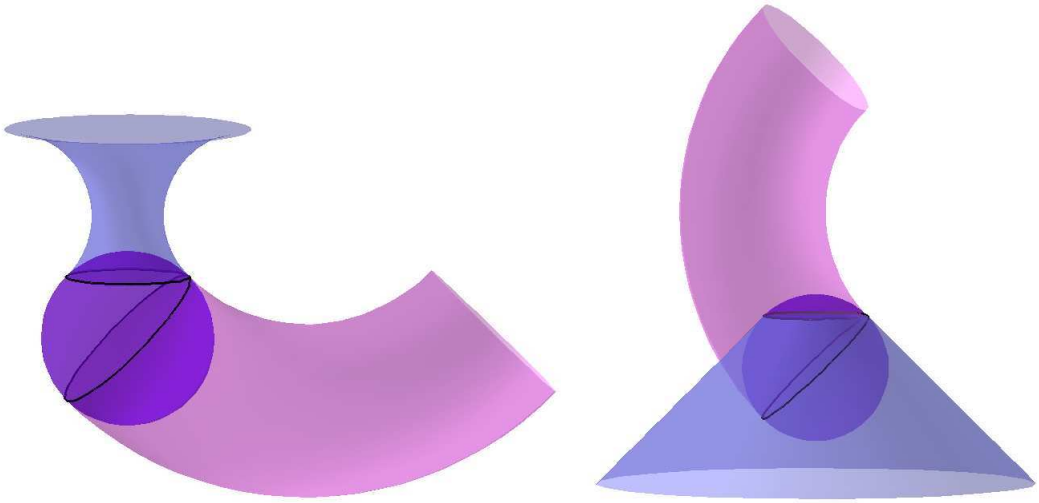


Figure 10: Two intersecting characteristic circles

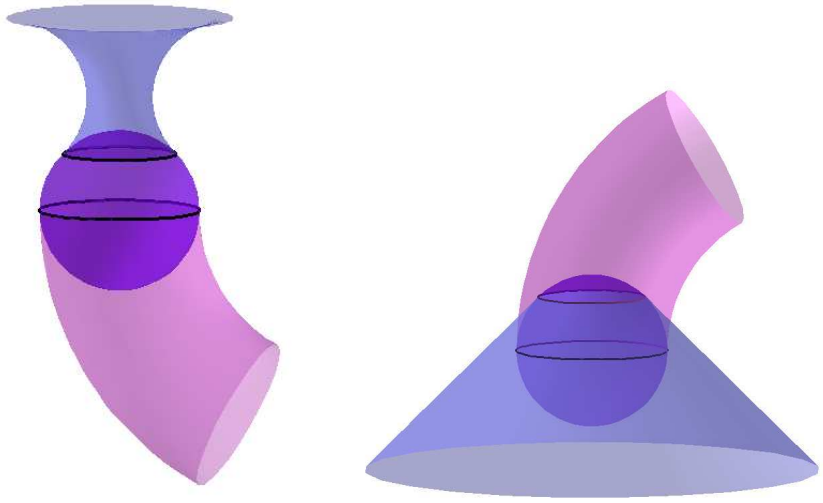
Now, let us see how to distinguish these different situations looking at the two arcs in the space of spheres.

In Subsection 3.1, we explained that the relative position of two spheres Σ_1 and Σ_2 can be determined from the position of the two corresponding points σ_1 and σ_2 in Λ^4.



(a) (b)

Figure 11: (a) An example of G^1 blend between two canal surfaces along two circles on a sphere without singularities. (b)The blend using the two other parts of the two canal surfaces form an enveloppe surface with singularities.



(a) (b)

Figure 12: (a) Another example without singularities. (b) Another example with singularities.

Here we are interested in the circles of the sphere Σ. Oriented circles of a sphere form a 3-dimensional quadric $\Lambda^3 \subset \mathbb{R}^4_1$. The construction is the same as the construction of the set of spheres in $\mathbb{R}^3$ we presented in Subsection 3.1.

The circles of Σ are the intersection of Σ with spheres orthogonal to Σ. The corresponding points of Λ^4 are the points of $\Lambda^3(\sigma) = (\mathbb{R} \cdot \sigma)^\perp \cap \Lambda^4$ (see Figure 13). Notice that $(\mathbb{R} \cdot \sigma)^\perp$ is parallel to the hyperplane $T_\sigma \Lambda^4$.

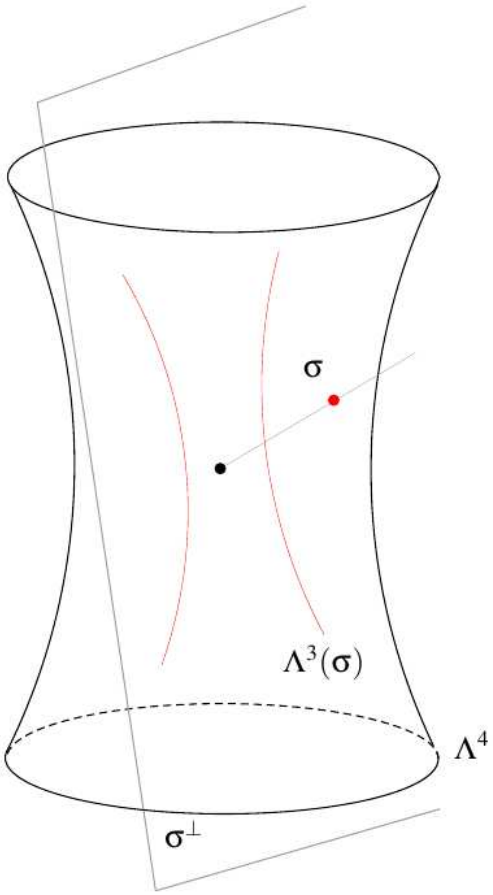


Figure 13: Representation of the space of circles on a sphere in Λ^4.

We can suppose that the two arcs $\gamma_1 : [a_1, b_1] \to \Lambda^4$ and $\gamma_2 : [a_2, b_2] \to \Lambda^4$ are parameterized by arc-length. The unit vectors $\dot{\gamma}_1(b_1)$ and $\dot{\gamma}_2(a_2)$ are tangent at $\sigma = \gamma_1(b_1) = \gamma_2(a_2)$ respectively to γ_1 and γ_2 (see Figure 14). Let Γ_1 and Γ_2 be the corresponding canal surfaces. The "last" characteristic circle C_1 of Γ_1 and the "first" characteristic circle C_2 of Γ_2 belong to the same sphere Σ which corresponds to the common point σ of γ_1 and γ_2.

The vector $\dot{\gamma}_1(b_1)$ can be seen as a point of Λ^4. In fact, it belongs to $\Lambda^3(\sigma)$. It corresponds to the sphere orthogonal to Σ along C_1. This amounts to say that we also consider the points of $\Lambda^3(\sigma)$ as circles of Σ. The points of the two curves $\dot{\gamma}_1(t)$ and $\dot{\gamma}_2(t)$ correspond to the derivated spheres $\dot{\Sigma}_i(t)$ which are orthogonal to the spheres corresponding to the points $\gamma_i(t)$; the intersection $\dot{\Sigma}_i(t) \cap \Sigma_i(t)$ is the characteristic circle of Γ_i contained in Σ_i. The two circles are disjoint if the point $\dot{\gamma}_2(a_2)$ is in the

"interior" of one of the cones $C_{\dot{\gamma}_1(b_1)}$ or $C_{-\dot{\gamma}_1(b_1)}$. Then, we have to determine the "good" part of these cones, that is to say the side where the points correspond to oriented circles C_1 and C_2, boundaries of two discs D_1 and D_2, $D_1 \subset D_2$ (that is to say $D_1 \cap D_2 = D_1$).

To do that, we will use the geodesic curvature vectors of each of the curves γ_1 and γ_2 noted $\overrightarrow{k_{g_1}}$ and $\overrightarrow{k_{g_2}}$.

The position of the two pieces of canal surfaces with respect to the sphere Σ associated to $\sigma = \gamma_1(b_1) = \gamma_2(a_2)$ is "good" if the "last" characteristic circles of the end of Γ_1 and the "first" characteristic circles of the beginning of Γ_2 project on circles of Σ disjoint from the annulus of Σ bounded by $C_1 \cup C_2$. The projections of the characteristic circles will be the intersection with Σ of the spheres corresponding to the points of $\dot{\gamma}_1(t)$ and of $\dot{\gamma}_2(t)$. The two curves $\dot{\gamma}_i(t)$ are on Λ^4 and not in $\Lambda^3(\sigma)$. As we want to work in $\Lambda^3(\sigma)$, we will consider a projection of these curves on $\Lambda^3(\sigma)$, $proj(\dot{\gamma}_i(t)) = \alpha_i(t)$. Then, the intersection with Σ of the sphere corresponding to $\alpha_i(t)$) is the circle $\Sigma \cap \dot{\Sigma}_i(t)$; along this circle, the two spheres intersect orthogonally.

Consider the point σ and the point $\dot{\gamma}_i(t)$. They corresponds to two spheres which are in a pencil of spheres with base circle $\dot{\Sigma}_i \cap \Sigma$. In Λ^4, the points σ and $\dot{\gamma}_i(t)$ are in a geodesic circle which cuts orthogonally $\Lambda^3(\sigma)$ at the point $\alpha_i(t)$. This defines the projection wee need [3]. Let the curves α_i be the images of the curves γ_i through this projection. These curves are time-like curves in $\Lambda^3(\sigma)$. At the extremities $\alpha_1(b_1)$ and $\alpha_2(a_2)$ of these curves, the tangent vectors to α_1 and α_2 are the geodesic curvature vectors $\overrightarrow{k_{g_1}}$ and $\overrightarrow{k_{g_2}}$ (the geodesic curvature vectors $\overrightarrow{k_{g_i}}$ are the projections of the vector $\ddot{\gamma}_i$ on $T_\sigma\Lambda^4$). First, the oriented circles associated to $\alpha_1(b_1)$ and $\alpha_2(a_2)$ where the canal join the sphere Σ are "good", if they belong to a pencil of spheres with limit points (time like geodesic curve in Λ^4), that is if they are disjoint. Then we need to be able to join the points $\alpha_1(b_1)$ and $\alpha_2(a_2)$ by a time-like curve in Λ^3. For that, the vector $\overrightarrow{\alpha_1(b_1)\alpha_2(a_2)}$ has to be a time-like vector. Moreover we want to join $\alpha_1(b_1)$, the end of the curve α_1 to the beginning $\alpha_2(a_2)$ of the curve α_2 by a time-like curve to obtain a curve where the time-variable x_0 is monotonous. Therefore the three vectors $\overrightarrow{k_{g_1}}$, $\overrightarrow{k_{g_2}}$ and $\overrightarrow{\alpha_1(b_1)\alpha_2(a_2)}$ must be on the same side of the cone $C_{\dot{\gamma}_1(b_1)}$.

Of course if at the extremity of γ_1 and at the initial point of γ_2 the tangent vectors are the same, the two canal surfaces are joined (G^1) along a characteristic circle.

In particular, suppose that we want to blend a canal surface with a piece of Dupin cyclide along a characteristic circle. We just have to consider the sphere S_0 of the canal surface containing the blending circle C_0. A point $\sigma \in \Lambda^4$ and a space-like vector $\dot{\sigma}$ tangent at σ to Λ^4 determine a circle on the sphere Σ associated to the point σ. The affine planes containing σ and $\dot{\sigma}$ which intersect Λ^4 in a space-like curve determine a Dupin cyclide. To the 3-parameter family of such planes H corresponds a 3-parameter family of such cyclides. We do not accept the vectorial plane containing σ and $\dot{\sigma}$, as it defines only the characteristic circle contained in Σ.

Therefore, we get a 3-parameter family of cyclides that can be blended with a canal surface along a characteristic circle. The figures 15 and 16 show two Dupin cyclides tangent along a characteristic circle.

The family of spheres tangent to a Dupin cyclide along a family of characteristic circles defined by $\theta = c^{te}$ contains two planes. The other family of spheres tangent along the other family of characteristic circles does not contain any plane. The planes can be seen in Λ^4 as the intersection of Λ^4 with the hyperplane $x_0 = x_4$ (see [D-G-L-M-B]). Therefore, we can distinguish the two brother circles (parameterized

[3] The circles $Aff(\sigma, \tau) \cap \Lambda^4$, where τ describes $\Lambda^3(\sigma)$, form a foliation of the region $(\{|\mathcal{L}(., \sigma| \leq 1\} \cap \Lambda^4) \setminus \{\sigma \cup (-\sigma)\}$ the leaves of which arrive orthogonally on $\Lambda^3(\sigma)$. The arcs cut on these circles by σ and $-\sigma$ define a projection of $(\{|\mathcal{L}(., \sigma| \leq 1\} \cap \Lambda^4) \setminus \{\sigma \cup (-\sigma)\}$ on $\Lambda^3(\sigma)$

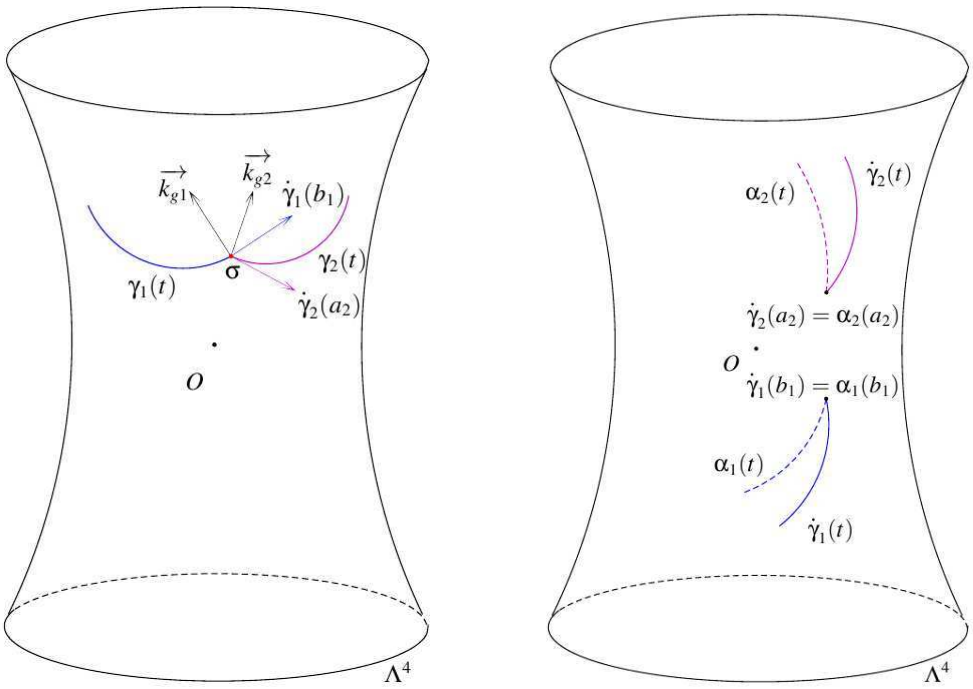


Figure 14: Representation of the problem in Λ^4; the dotted lines $\alpha_i(t)$ are the projections of the curves $\dot{\gamma}_i(t)$ on $\Lambda^3(\sigma)$.

respectively by θ and ψ) using this hyperplane: if the intersection of the hyperplane and the 2-plane containing a brother circle of a cyclide is not empty, it corresponds to the family of spheres tangent to the cyclide along characteristic circles defined by $\theta = c^{te}$.

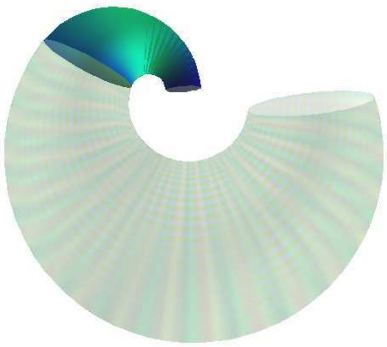


Figure 15: Two parts of Dupin cyclides tangent along a characteristic circle, on each cyclide the characteristic circle is obtained by a constant value of θ in the parametric equation.

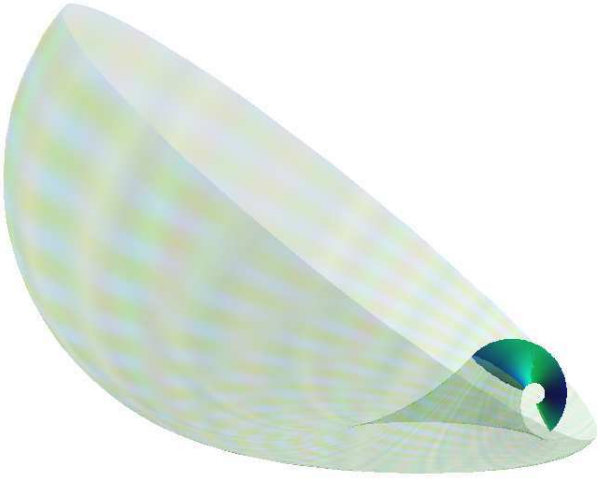


Figure 16: Two parts of Dupin cyclides tangent along a characteristic circle, on one cyclide the circle is obtained by a constant value of θ whereas, on the other cyclide, it is obtained by a constant value of ψ.

Choosing one more sphere $\tau \in \Lambda^4$, the triple $\sigma, \dot{\sigma}, \tau$ determines an affine plane H (the point τ cannot be of the form $\sigma + \lambda\dot{\sigma}$ as $\dot{\sigma}$ is space-like). If the intersection $H \cap \Lambda$ is space-like it provides a cyclide tangent to σ along the circle $\sigma^\perp \cap \dot{\sigma}^\perp$ and tangent to τ along a circle. This cyclide may have a singular point (see Figure 17).

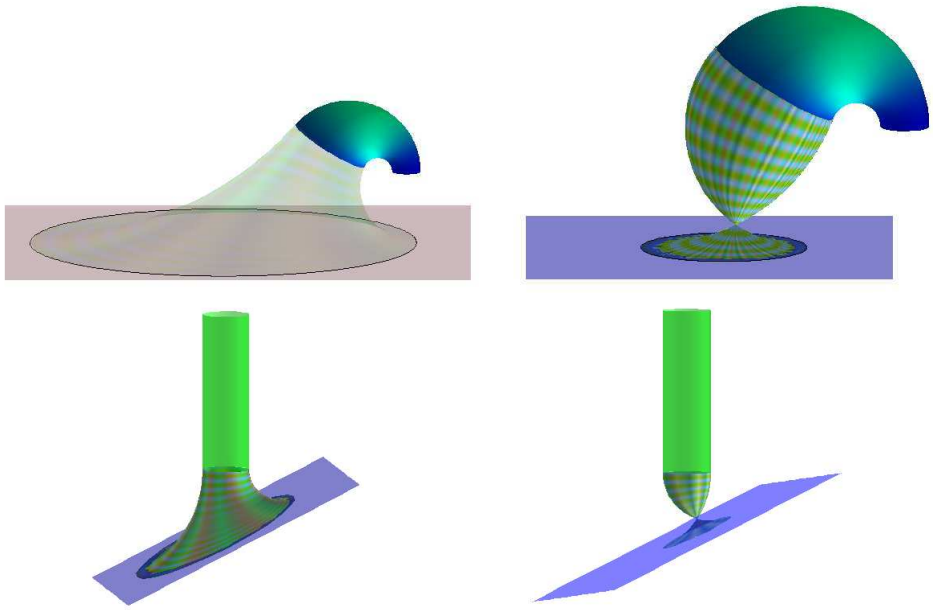


Figure 17: Examples of blend between a canal surface and a plane using a part of Dupin cyclide.

As two pairs $(\sigma, \dot{\sigma})$ and $(\tau, \dot{\tau})$ are not, in general, contained in an affine plane, one cannot, in general, join two pieces of canal surfaces by one piece of Dupin cyclide. Using two pieces, the join is possible (see Figure 18; Boehm [Bo] studies a similar problem directly in $\mathbb{R}^3$).

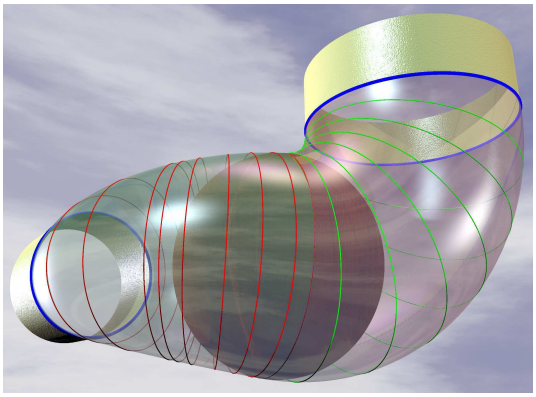
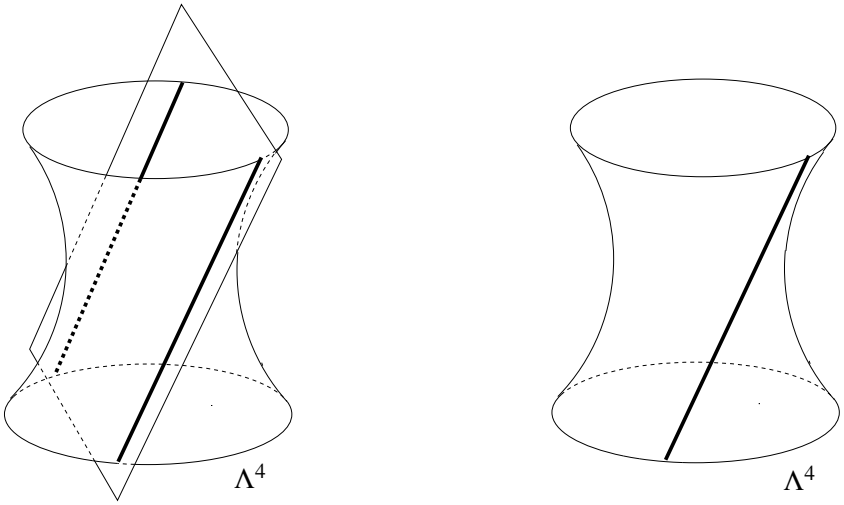


Figure 18: Example of blend between two canal surfaces using two parts of Dupin cyclides.

5 Contact condition

A contact condition in $\mathbb{R}^3$ is a pair, a point m and a plane h contained in the 3-dimension tangent space to the ambient space, $(m, h \subset T_m\mathbb{R}^3)$. It defines a pencil of spheres: the spheres tangent to h at m. We have seen that such a pencil is represented by two parallel light-rays, and if we specify the orientation of the spheres, one light-ray only (see Figure 19).



Two rays, the two orientations

One ray, one orientation is chosen

Figure 19: Tangent spheres at a point m

Given a surface M of class C^2, the spheres tangent to M at a point m form a light-ray.

The points of Λ^4 representing spheres tangent to a surface M is a 3-dimensional set $W(M)$. It is singular at the osculating spheres which form in general a 2-dimensional set. Let C be a curve drawn on M, then the points corresponding spheres tangent to M along C form a two-dimensional object $W(C)$ that we will meet again.

When the surface M is a canal surface (see Subsection3.3), one of the components of the points representing osculating spheres is the curve γ generating the canal.

Recall that when the surface is a Dupin cyclide, it is in two different ways the envelope of a one parameter family of spheres. The osculating spheres to the Dupin cyclide are now the two corresponding curves C_P and $C_{P^*} = P^* \cap \Lambda^4$.

The unit tangent vector $\dot{\sigma}$ at a point $\sigma \in C_P$ corresponds to a sphere which intersects the sphere Σ corresponding to σ along a characteristic circle of the cyclide (see Figure 7).

The points corresponding to spheres tangent to the Dupin cyclide C is the union of the lines, which are light-rays, joining a point of C_P to a point of C_{P^*}. It is a 3-dimensional subset $W(C)$ of Λ^4.

Remark:

Given a Villarceau circle $\Gamma \subset C$ we get a map from Γ to the pencil $\mathcal{P}_\Gamma$ of spheres of axis Γ, which maps a point $m \in \Gamma$ to the sphere Σ_m of the pencil which is tangent to C at the point.

We get a plane $h_m = T_mC = T_m\Sigma_m \subset T_m\mathbb{E}^3 \subset T_m\mathcal{L}ight$ at each point $m \in \Gamma$, and therefore a light-ray ℓ_m composed of the spheres tangent at m to the cyclide C. The union of these light-rays form a

surface $\mathcal{S}$ contained in Λ^4. One can prove that parallel affine hyperplanes orthogonal to δ intersect $\mathcal{S}$ in circles which correspond to a one parameter family of cyclides tangent to C along γ. This result we be a consequence of our general result Theorem 8.0.2. It can also be proved using complex geometry.

5.1 Three contact conditions

Let us consider three contact data $(m_1, h_1); (m_2, h_2); (m_3, h_3)$, where the three oriented planes h_1, h_2, h_3 are in $T_{m_1}\mathbb{R}^3, T_{m_2}\mathbb{R}^3, T_{m_3}\mathbb{R}^3$. We will use the three light-rays ℓ_1, ℓ_2 et ℓ_3 in the de Sitter space Λ^4 formed by spheres through m_1 and tangent to h_1, through m_2 tangent to h_2, and through m_3 tangent to h_3 (the spheres inherit their orientations from the planes h_1, h_2 or h_3).
Remark: Let us consider three points m_1, m_2, m_3 of a Dupin cyclide C. An orientation of the Dupin cyclide determines the three light-rays ℓ_1, ℓ_2, ℓ_3. Let us replace ℓ_1 by $-\ell_1$. Then C does not satisfying the new set of three contact conditions. We will later give examples where a cyclide, different from C satisfies the three contact conditions (see Figure 31).

The dimension of the set of three points on a Dupin cyclide is $15 = 9 + 2 + 2 + 2$ (9 is the dimension of the affine Grassmann manifold of planes in $\mathbb{R}^5_1$, therefore the dimension of the set of cyclides). The dimension of the set of triple contact conditions in $\mathbb{R}^3$ is $15 = 3(3 + 2)$, as $(3 + 2)$ is the dimension of the set of pairs (m, a plane direction at m). We may expect that three contacts problems admits a finite number of solution (or none). The situation is quite different.

We can chose one sphere σ_i in each light-ray ℓ_i. This defines an affine plane P. The intersection $P \cap \Lambda^4$ is a curve C_P. If this curve C_P is space-like, the corresponding envelope is a Dupin cyclide. In general it does not satisfy the given contact condition, as the characteristic circle Γ_i contained in the sphere Σ_i needs not to contain the point m_i corresponding to the light-ray ℓ_i. If the point m_i belongs to the characteristic circle $\Gamma_i \subset \Sigma_i$, the cyclide satisfies contact condition at m_i and the light-ray ℓ_i intersects the brother affine plane P^*.

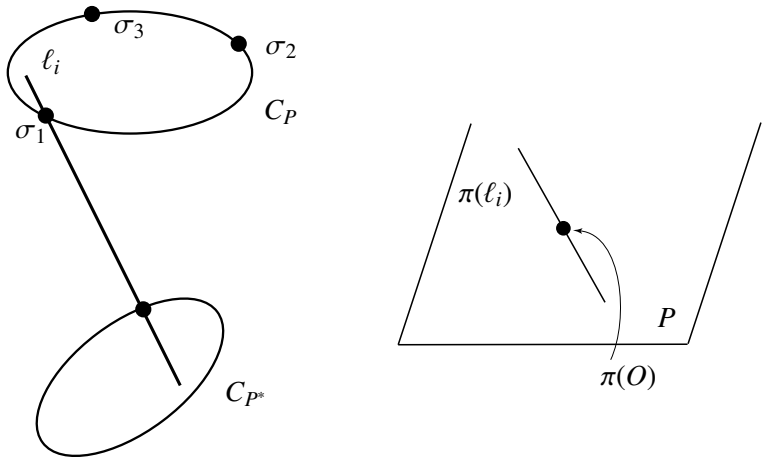


Figure 20: The projections of a light-ray ℓ_i contained in Λ^4 when the cyclide satisfies the three contact conditions

Recall that p denotes the vector space parallel to P, p^* the vector space parallel to P^* and δ the line $(p \oplus p^*)^\perp$. Let π be the orthogonal projection on P (the kernel is the space $p^* \oplus \delta$). For an arbitrary

choice of $\sigma_i \in \ell_i$, the light-rays ℓ_i do not cross P^*. Therefore the projection of a light-ray ℓ_i in general does not contain the point $\pi(O)$ of P, orthogonal projection on P along $p^* \oplus \delta$ of the origin O of $\mathbb{R}^5_1$, that we will take as origin of P. If the "'circle" $C_P = P \cap \Lambda^4$ defines a cyclide tangent to the three contact conditions, it does contain the origin, as in this case each of the light-ray crosses $C_P^* = P^* \cap \Lambda^4$ at a point which projects on the origin $\pi(O)$. It is not a priori clear that such an affine plane P exists. We will prove that, to guarantee the existence of an affine plane $P = Aff(\sigma_1, \sigma_2, \sigma_3); \sigma_i \in \ell_i$ the contact conditions should satisfy a condition. A necessary condition is that the light-rays ℓ_i should project orthogonally on P (parallel to $p^* \oplus \delta$) into lines containing the origin $\pi(O)$ of P (see figure 20). More work would be necessary to prove that the conditions obtained are also sufficient to guarantee the existence of a Dupin cyclide satisfying the three contact conditions.

We will not perform the computations involved as another, more dynamical, viewpoint will provide more easily a necessary and sufficient condition which guarantees the existence of a cyclide satisfying the three contact conditions.

6 Find a Dupin cyclide satisfying three contact conditions

6.1 The homographies pang, pong and ping

In the previous sections, we were dealing with Möbius geometry an the space of oriented spheres. Now we will need another tool: projective geometry.

An *homography* $\Phi : \mathbb{R} \mapsto \mathbb{R}$ is defined by the formula $x_2 = \frac{ax_1+b}{cx_1+d}$, where a, b, c, d are real numbers such that $det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$. Let us consider the line $\mathbb{R}$ as the affine line $y = 1$ of $\mathbb{R}^2$. The invertible linear map of matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ maps the line (O, x_1) to the line (O, x_2) whenever the fraction $x_2 = \frac{ax_1+b}{cx_1+d}$ has a finite value. It is convenient to write $x_2 = \infty$ when $x_2 = \frac{ax_1+b}{cx_1+d}$ is not defined. The image of the point ∞ is the point $x_2 = \frac{a}{b}$. In this way we are extending the homographies to continuous bijection of the projective line $\mathbb{P}^1 = \mathbb{R} \cup \infty$ onto itself. The homographies are also characterized by the fact they preserve cross-ratios.

Notice that, if the invertible linear map of matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ exchanges two lines, its matrix in a basis formed of vectors of these two lines becomes of the form $\begin{pmatrix} 0 & b' \\ c' & 0 \end{pmatrix}$. Observe that the latter matrix is of trace zero. Then $trace\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ was already null. The square of the matrix $\begin{pmatrix} 0 & b' \\ c' & 0 \end{pmatrix}$ is $\begin{pmatrix} b'c' & 0 \\ 0 & b'c' \end{pmatrix}$. The corresponding homography is therefore the identity.

Reciprocally, an invertible linear map of zero trace which has no fixed line has eigenvalues of the form $\pm\rho i, \rho \in \mathbb{R}^+$. In the plane endowed with the Euclidean metric such that the basis is orthonormal, it is the composition of a rotation of angle $\pi/2$ or $-\pi/2$ and a homothety. The corresponding homography is therefore an involution exchanging pairs of lines.

Homographies between two affine lines ℓ_1 and ℓ_2 contained in some vector space $\mathbb{R}^2$ are obtained considering the intersection of these two lines with a pencil of affine hyperplanes $\mathcal{P}$ of $\mathbb{R}^n$ such that all the hyperplanes of the pencil are transverse to the two lines; that way we get a map $\ell_1 \to \ell_2$ defined whenever the hyperplane $P \in \mathcal{P}$ of the pencil intersects transversally ℓ_1 and ℓ_2.

Quadratic hyperboloids of $\mathbb{R}^3$ contain two families of lines. Given two lines ℓ_1 and ℓ_2 of one family, the lines of the other family provide a homography $\varphi : \ell_1 \to \ell_2$. The proof of the latter fact relies on classical geometry of quadrics ([Cha]).

It is not surprising that the quadric $\Lambda^4 \subset \mathbb{R}^5_1$ also provides homographies between disjoint affine light-rays.

Here we obtain a homography using a contained in Λ^4 construction slightly different from the two mentioned above. The lines ℓ_1 and ℓ_2 are disjoint affine light-rays contained in Λ^4.At each point $\sigma \in \ell_i$, we consider the affine tangent hyperplane $T_\sigma\Lambda^4$ to Λ^4 at σ. We suppose that the light-ray ℓ_2 is not contained in any of the hyperplanes $T_\sigma\Lambda^4$. Let us first define the map *pang* at the points $\sigma_1 \in \ell_1$ such that $T_{\sigma_1}\Lambda^4 \cap \ell_2 \neq \emptyset$

$$pang : \ell_1 \to \ell_2;\ pang(\sigma_1) = T_{\sigma_1}\Lambda^4 \cap \ell_2 \tag{3}$$

Let us call $\hat{\ell}_1 = \ell_1 \cup \infty$ the projective completion of ℓ_1 and $\hat{\ell}_2 = \ell_2 \cup \infty$ the projective completion of ℓ_2. We can extend the previous *pang* map to the projective completions of ℓ_1 and ℓ_2, and still call it *pang*, by

$$\begin{gathered} pang : \hat{\ell}_1 \to \hat{\ell}_2;\ pang(\sigma_1) = T_{\sigma_1}\Lambda^4 \cap \ell_2 \text{ if } T_{\sigma_1}\Lambda^4 \cap \ell_2 \neq \emptyset \\ pang(\sigma_1) = \infty \text{ if } T_{\sigma_1}\Lambda^4 \cap \ell_2 = \emptyset,\ \ pang(\infty) = lim_{\sigma\to\infty} T_{\sigma_1}\Lambda^4 \cap \ell_2. \end{gathered} \tag{4}$$

Proposition 6.1.1. *The map pang is a homography.*

Proof: Two disjoint light-rays $\ell_1 \subset \Lambda^4$ and $\ell_2 \subset \Lambda^4$ span a 3-dimensional affine space Q which, because it contains two independent light directions, is time-like. The intersection $Q \cap \Lambda^4$ is a 2-dimensional quadratic hyperboloid $\tilde{Q}^2$. The tangent hyperplane at $\sigma_1 \in \ell_1$ to Λ^4 intersects Q in the tangent plane at σ_1 to $\tilde{Q}^2$; this tangent plane intersects $\tilde{Q}^2$ in two lines, ℓ_1 and another line $\ell_{12}(\sigma_1)$. Completing the five-dimensional space $\mathbb{R}^5_1$ in a projective space $\mathbb{RP}^5$, Q is contained in the 3-dimensional projective space $\hat{Q} \simeq \mathbb{RP}^3$, projective completion of Q, and the light-rays ℓ_i and $\ell_{12}(\sigma_1)$ are completed into projective lines $\hat{\ell}_i$ and $\hat{\ell}_{12}(\sigma_1)$. Consider now a point $\sigma_1 \in \hat{\ell}_1$. In this projective context, let us denote by PT the projective subspace tangent to a surface contained in a projective space. The intersection $\hat{\tilde{Q}}^2 \cap PT_{\sigma_1}\hat{\tilde{Q}}^2$ contains the two projective lines: $\hat{\ell}_1$ and $\hat{\ell}_{12}(\sigma_1)$. The projective line $\hat{\ell}_{12}(\sigma_1)$ intersects $\hat{\ell}_2$ at a point $pang(\sigma_1)$. As the family of lines $\hat{\ell}_{12}(\sigma_1)$ is the second family of projective lines ruling $\hat{\tilde{Q}}^2$, using Chasles work, we see that the map *pang* is a homography. □

Let us define now two other maps

$$\begin{aligned} pong : \ell_2 \to \ell_3;\ pong(\sigma_2) = T_{\sigma_2}\Lambda^4 \cap \ell_3 \\ ping : \ell_3 \to \ell_1;\ ping(\sigma_3) = T_{\sigma_3}\Lambda^4 \cap \ell_1 \end{aligned} \tag{5}$$

As we did for the map *pang* (see Formula 3 and 4), we can extend the maps *pong* and *ping* to the projective completions $\hat{\ell}_2$ of ℓ_2, and $\hat{\ell}_3$ of ℓ_3 respectively.

6.2 Objects naturally associated to three contact conditions

A circle and six spheres are naturally associated to the situation. Through the three points m_1, m_2, m_3 passes a circle-or-line $\Gamma_{123} \subset \mathbb{R}^3$. Among the oriented spheres tangent to the given oriented plane $h_1 \subset T_{m_1}\mathbb{R}^3$, one, $\sigma_{11,2}$ passes through m_2 and one $\sigma_{11,3}$ passes through m_3. Similarly, we can define two oriented spheres σ_{221} and σ_{223} tangent to h_2 at m_2 and two oriented spheres σ_{331} and σ_{332} tangent to h_3 at m_3.

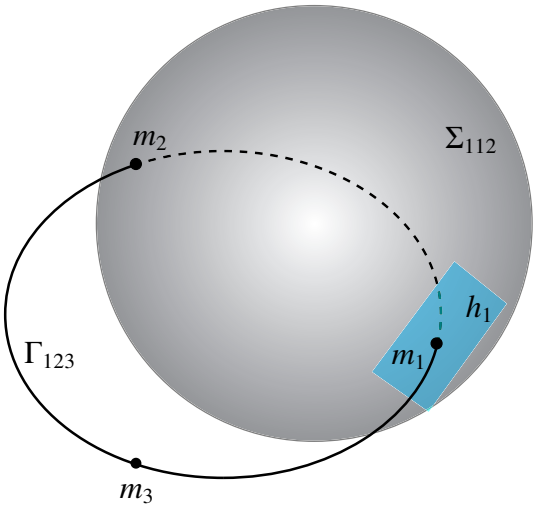


Figure 21: The circle Γ_{123}, the three points m_1, m_2, m_3 and the sphere Σ_{112} tangent at m_1 to the plane h_1

6.3 The *ping* ∘ *pong* ∘ *pang* map

In order to turn computations easier, we will chose
- on the light-ray ℓ_1, the origin σ_{112}, the point corresponding to the sphere which also contains m_2,
- on the light-ray ℓ_2, the origin σ_{223}, the point corresponding to the sphere which also contains m_3,
- on the light-ray ℓ_3, the origin σ_{331}, the point corresponding to the sphere which also contains m_1.

We will also chose the light-like vectors m_i such that

$$\mathcal{L}(m_i, m_j) = -1. \tag{6}$$

Then, in a model $\mathcal{H} \cap Light$ of the Euclidean space containing the three points m_i, they form an equilateral triangle inscribed in Γ_{123} the sides of which are of length 2. We will use the same notation for the points $m_i \subset \mathbb{R}^3$, and for the light vectors contained in the model $\mathcal{H} \cap Light$ (recall that $\mathcal{H}$ is an affine hyperplane of $\mathbb{R}^5_1$; to chose it is equivalent to the choice of the metric on $\mathbb{R}^3$ (see Figure 3)).

Points on ℓ_1 are of the form $\sigma_1 = \sigma_{112} + k_1 m_1$, points on ℓ_2 are of the form $\sigma_2 = \sigma_{223} + k_2 m_2$ and points on ℓ_3 are of the form $\sigma_3 = \sigma_{331} + k_3 m_3$.

The point $pang(\sigma_1)$, as it belongs to the affine hyperplane $T_{\sigma_1}\Lambda^4$ of equation $\mathcal{L}(\sigma_1, \sigma) = 1$, has coordinate k_2 solution of the equation

$$\mathcal{L}(\sigma_{112} + k_1 m_1, \sigma_{223} + k_2 m_2) = 1$$

Therefore

$$k_2 = \frac{-1 + \mathcal{L}(\sigma_{223}, \sigma_{112}) + k_1 \mathcal{L}(\sigma_{223}, m_1)}{k_1}. \tag{7}$$

In the same way we compute the coordinate k_3 of $pong(\sigma_2)$ and k_1 of $ping(\sigma_3)$

$$\begin{aligned} k_3 &= \frac{-1 + \mathcal{L}(\sigma_{331}, \sigma_{223}) + k_2 \mathcal{L}(\sigma_{331}, m_2)}{k_2} \\ k_1 &= \frac{-1 + \mathcal{L}(\sigma_{112}, \sigma_{331}) + k_3 \mathcal{L}(\sigma_{112}, m_3)}{k_3}. \end{aligned} \tag{8}$$

Let

$$\begin{cases} \alpha_1 = \mathcal{L}(\sigma_{112}, m_3) \\ \alpha_2 = \mathcal{L}(\sigma_{223}, m_1) \\ \alpha_3 = \mathcal{L}(\sigma_{331}, m_2) \\ \beta_1 = -1 + \mathcal{L}(\sigma_{223}, \sigma_{112}) \\ \beta_2 = -1 + \mathcal{L}(\sigma_{331}, \sigma_{223}) \\ \beta_3 = -1 + \mathcal{L}(\sigma_{112}, \sigma_{331}) \end{cases} \tag{9}$$

Notice also that the four terms α_i and β_i are conformal invariant of the picture determined by the three oriented contact conditions.

We have seen that the three oriented contact conditions determine one circle Γ_{123} and six spheres. Here we have selected three of the six oriented spheres as origin on the light-rays ℓ_i. As any pair taken among the six spheres intersect, the terms β_i are of the form $-1 + cos\theta$, where θ is the angle between the two spheres involved. The term α_i is a function of the angle of the circle Γ_{123} and one of the spheres.

We can now write matrices corresponding to the homographies $pang : \hat{\ell}_1 \to \hat{\ell}_2$, $pong : \hat{\ell}_2 \to \hat{\ell}_3$ and $ping : \hat{\ell}_3 \to \hat{\ell}_1$

$$pang = \begin{pmatrix} \alpha_2 & \beta_1 \\ 1 & 0 \end{pmatrix} \quad pong = \begin{pmatrix} \alpha_3 & \beta_2 \\ 1 & 0 \end{pmatrix} \quad ping = \begin{pmatrix} \alpha_1 & \beta_3 \\ 1 & 0 \end{pmatrix} \tag{10}$$

On Figure 22, $\sigma'_2 = pang(\sigma_1)$, $\sigma_3 = pong(\sigma'_2)$, $\sigma'_1 = ping(\sigma_3)$, $\sigma_2 = pang(\sigma'_1)$, $\sigma'_3 = pong(\sigma_2)$ and finally $ping(\sigma'_3) = \sigma_1$.

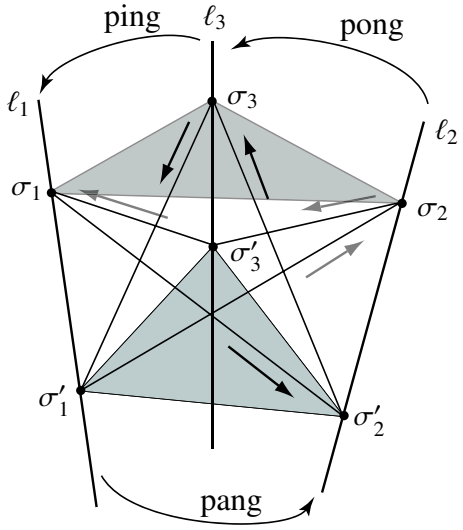


Figure 22: $ping \circ pong \circ pang$ map

If the oriented contacts are tangent to a cyclide, then the composition $ping \circ pong \circ pang$ need to exchange the two points of intersection of ℓ_1 with the brother circles (see Figure 22). A homography which exchanges two points have to be an involution different from the identity.

The corresponding linear map is, up to a factor $c \cdot Id$, an involution of $\mathbb{R}^2$. We have seen in Subsection 6.1 that an invertible linear map of $\mathbb{R}^2$ induces an involution different from the identity if and only if it has trace zero.

The trace of the product $ping \circ pong \circ pang$ is

$$\mathcal{A} = \alpha_1\alpha_2\alpha_3 + \beta_3\alpha_2 + \beta_2\alpha_1 + \beta_1\alpha_3 \tag{11}$$

So we get the necessary condition for the contact conditions to be tangent to a cyclide

$$\mathcal{A} = \alpha_1\alpha_2\alpha_3 + \beta_3\alpha_2 + \beta_2\alpha_1 + \beta_1\alpha_3 \ = \ 0 \tag{12}$$

Remark: It is now natural to replace the whole Lorentz space $\mathbb{R}^5_1$ by its projective completion $\mathbb{RP}^5$. This projective space is the space of lines of $\mathbb{R}^6$. The 6-dimensional space is now endowed with a definite quadratic form of signature 2. The intersection of the light-cone of the latter quadratic form an the affine chart $\mathbb{R}^5_1$ is the quadric Λ^4. The geometry induced on $\mathbb{RP}^6$ by linear isomorphisms of $\mathbb{R}^6$ preserving the light cone is called Lie sphere geometry. A detailed reference is [Ce]. The quadric Λ^4 is completed in a smooth quadric. The lines, hyperplanes and planes are completed into projective lines, hyperplanes and planes. Therefore the maps $pang$, $pong$, $ping$ and their inverses admit projective completions. The quadric Λ^4 is contained in a smooth projective quadric. Therefore the tangent hyperplanes $T_\sigma\Lambda^4$ to Λ^4 when σ goes to infinity on a light-ray have a limit in $\hat{\Lambda}^4 \subset \mathbb{RP}^5$. The fact the $ping \circ pong \circ pang$ map is an involution implies that for any $\sigma_1 \in \ell_1$, such that all the maps $pang$, $pong$, $ping$, $pang^{-1}$, $pong^{-1}$, $ping^{-1}$ and the maps obtained composing up to 6 of the previous maps do not send σ_1 "to infinity", the affine planes $Aff(\sigma_1, pong \circ pang(\sigma_1), pang \circ pong \circ ping \circ pang(\sigma_1))$ and $Aff(pang(\sigma_1), ping \circ pong \circ pang(\sigma_1), pong \circ pang \circ ping \circ pong \circ pang(\sigma_1))$ are brothers.

For example, let us prove that the $ping \circ pong \circ pang(\sigma_1)$ belongs to the intersection $T_{\sigma_1}\Lambda^4 \cap T_{pong\circ pang(\sigma_1)}\Lambda^4 \cap T_{pang\circ ping\circ pong\circ pang(\sigma_1)}\Lambda^4$.

- As $ping \circ pong \circ pang(\sigma)$ belongs to ℓ_1 it belongs to $T_{\sigma_1}\Lambda^4$.
- As $ping \circ pong \circ pang(\sigma_1) = ping(pong \circ pang(\sigma_1))$, it belongs to $T_{pong\circ pang(\sigma)}\Lambda^4$.
- As $ping\circ pong\circ pang(\sigma_1) = pang^{-1}(pang\circ ping\circ pong\circ pang(\sigma))$, it belongs to $T_{pang\circ ping\circ pong\circ pang(\sigma)}\Lambda^4$.

Remark: The cases where one product of less that 6 of the maps $ping$, $pong$, $pang$, $ping^{-1}$, $pong^{-1}$, $pang^{-1}$ send σ "to infinity" can be dealt with using an argument of continuity, extending the maps to $\hat{\ell}_1$, $\hat{\ell}_2$ and $\hat{\ell}_3$, and using the fact that the tangent hyperplanes $T_\sigma\Lambda^4$ have a limit when σ goes "to infinity" on a light-ray, is left to the reader.

It is proved in [L-W] that the brother P^* of the plane P it enough to is enough to take the intersection of the tangent hyperplanes to Λ^4 at three different points of $P \cap \Lambda^4$ and that the brother P of P^* is obtained taking the intersection of the tangent hyperplanes to Λ^4 at three different points of $P^* \cap \Lambda^4$. Therefore a pair $\sigma_1 \in \ell_1,\ pang \circ pong \circ ping(\sigma_1) \in \hat{\ell}_1$ determines, when the map $ping \circ pong \circ pang$ is an involution without fixed points, a pair of brother planes.

This give a one-parameter family of cyclides, as the initial point $\sigma_1 \in \hat{\ell}_1$ is arbitrary.

We have proven the following

Theorem 6.3.1. *Three contact conditions corresponding to three disjoint light-rays define a one parameter family of Dupin cyclides tangent to the three contact conditions if and only if the $ping \circ pong \circ pang$ map constructed from the light-rays is and involution, that is if and only if $\mathcal{A}(\ell_1, \ell_2, \ell_3) \ = \ 0$*

7 A geometric interpretation of the admissibility condition

The singular Dupin cyclide generated by the spheres Σ'_{113}, Σ'_{223} and the point m_3 (which is a singular point of the cyclide) is tangent to the two spheres Σ_{113} and Σ_{223}. As the brother family of spheres to the one containing the spheres Σ'_{113}, Σ'_{223} and the point m_3 contains already two spheres containing m_3 and

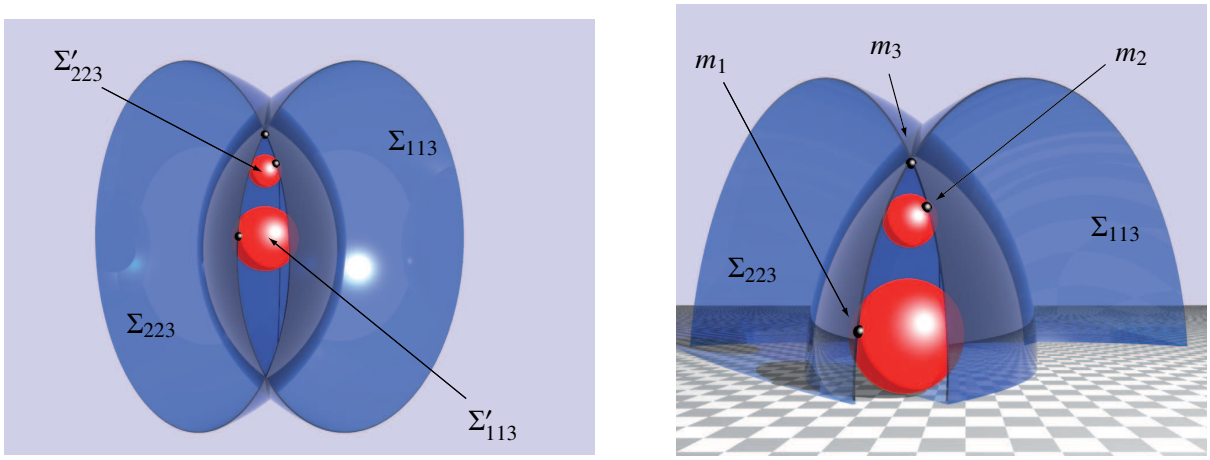


Figure 23: The two spheres which define a solution singular at m_3

tangent to Σ'_{113}, Σ'_{223} at m_1 and m_2, the three points m_1, m_2, m_3 are on the cyclide. Therefore it satisfies the requirement. The contact condition through m_3 tangent to the cone are satisfied. The equation $\mathcal{A} = 0$ guarantees that the admissible conditions containing the first two and the point m_3 form a circle, a point or is empty in $\mathbb{P}^*(T_{m_3}\mathbb{R}^3)$, the set of planes of $T_{m_3}\mathbb{R}^3$. It is therefore the circle of tangent plane to the cone tangent at m_3 to the singular cyclide we found above.

8 The common tangent curve

A particular case of Dupin cyclide which are tangent along a curve which is not a characteristic circle is provided by complex geometry.

Let $\mathbf{T}_{a,b} \subset \mathbb{S}^3 \subset \mathbb{C}^2$ be the torus of equations $|z_1{}^{-2} = a^2$, $|z_2|^2 = b^2$, $z_1 \in \mathbb{C}, z_2 \in \mathbb{C}$, $a \in \mathbb{R}, b \in \mathbb{R}, a^2 + b^2 = 1$. The action of the unit circle $\mathcal{U} = \{e^{i\theta}\} \subset \mathbb{C}$ on $\mathbb{C} \times \mathbb{C}$ globally preserves the sphere $\mathbb{S}^3$ and the torus $\mathbf{T}_{a,b}$. Its orbit are circles, in particular its orbits contained in $\mathbf{T}_{a,b}$ form one of the families of Villarceau circles. Consider a sphere Σ tangent to $\mathbf{T}_{a,b}$ at a point m. The orbit of the sphere Σ envelopes a torus which is tangent to $T_{a,b}$ along the whole Villarceau circle orbit of the point $m_i n T_{a,b}$. Using a stereographic projection on $\mathbb{R}^3$, we obtain cyclides tangent along a Villarceau circle (see Figure 24).

Let us show that Dupin cyclides maybe tangent along a more general family of curves.

Theorem 8.0.2. *Three contact conditions on a Dupin cyclide determine a homography Φ between the two conics C_P and C_{P^*}. The graph of this homography determines a curve Γ in the cyclide which is where all the cyclide solutions of the three contacts problem are tangent.*

Remark: We accept a contact at a singular point if the plane is tangent to the cone tangent to the cyclide at the singular point.

The light-rays joining one point of P and one point of P^* correspond to all contacts conditions obtained from a point of the cyclide C envelope of the spheres corresponding to the points of $C_P = P \cap \Lambda^4$. Therefore the homography Φ which defines a one-parameter family of light-rays, defines a curve on C. This curve is of homology class $(1, \pm 1) \in H^1(C)$ if the cyclide is regular and the generators of the homology of the cyclide are one characteristic circle in each family.

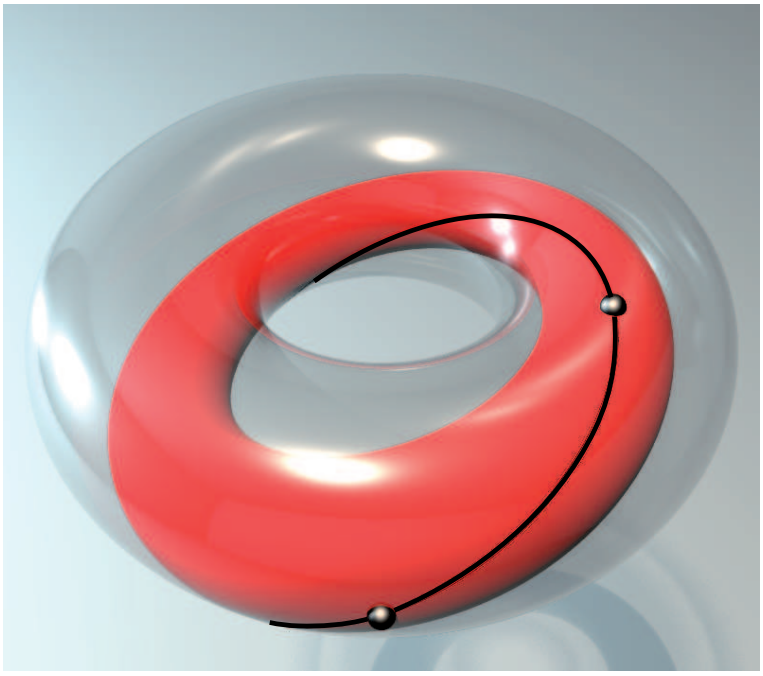


Figure 24: Cyclides tangent along a Villarceau circle

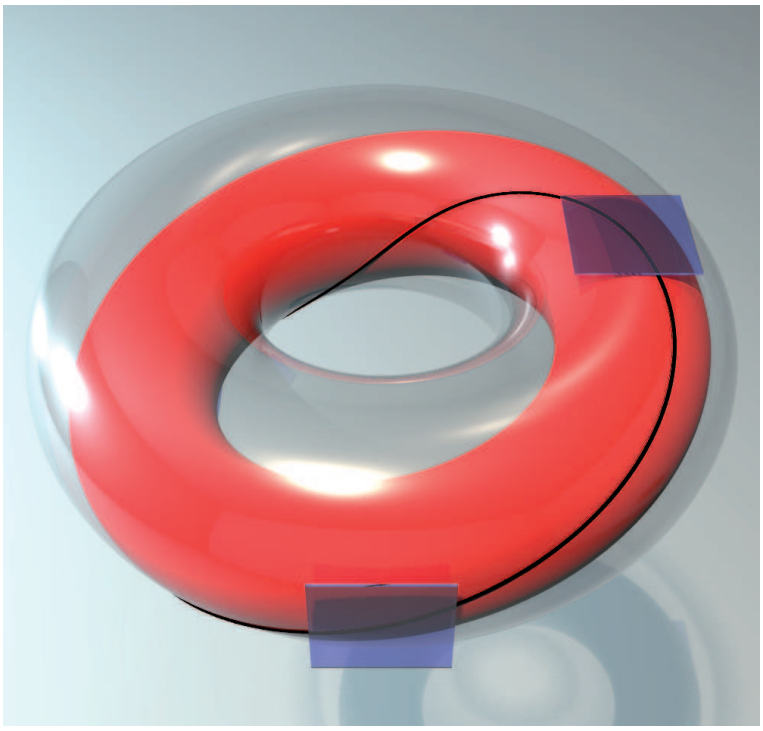


Figure 25: A torus and a tritangent cyclide

Remark: Let us rephrase our results using Lie sphere geometry (see Remark 6.3). The 6-dimensional space underlying the projective completion of the Lorentz space $\mathbb{R}^5_1$ can be endowed with a structure of Minkowski space of signature $(4,2)$. Let us consider the coordinates $x_{-1}, x_0, x_1 \cdots x_4$ on $\mathbb{R}^6_2$ endowed with the Minkowski form $\bar{\mathcal{L}}(x_{-1}, x_0, x_1 \cdots x_4) = -x_{-1}^2 - x_0^2 + x_1^2 + \cdots + x_4^2$; $\mathbb{R}^5_1$ is identified with the hyperplane of equation $x_{-1} = 1$ of $\mathbb{R}^6_2$.

We see that the quadric $\Lambda^4 \subset \mathbb{R}^5_1 \simeq \{x_{-1} = 1\}$ is now contained is the light-cone $\bar{\mathcal{L}}ight$ of equation $-x_{-1}^2 - x_0^2 + x_1^2 + \cdots + x_4^2 = 0$. For this pseudo-metric, the 3-dimensional spaces $\bar{P} = span(P)$ and $\bar{P}^* = span(P^*)$ are orthogonal.

The homography Φ is obtained from a linear map $F : \bar{P} \to \bar{P}^*$ (defined up to a constant factor).

Let now start with a point $\sigma_\lambda \in \ell_1$ which is a barycentric combination of $\sigma = P \cap \ell_1$ and $\sigma^* = P^* \cap \ell_1$, $\sigma_\lambda = \lambda\sigma + (1-\lambda)\sigma^*$.

We need to prove that all the cyclides of the one parameter family defined changing the initial point $\sigma_\lambda \in \ell_1$ are tangent to the contact data defined by a light-ray ℓ_τ joining a point $\tau \in \Lambda^4 \cap P_\sigma$ to the point $\Phi(\tau) \in P^*_\sigma$.

For that, let us first remark that the affine planes

$$P_\lambda = Aff(\sigma_\lambda, pong \circ pang(\sigma_\lambda), ping \circ pong \circ pang \circ ping(\sigma_\lambda))$$

are the intersection of 3-dimensional vector spaces $\bar{P}_\lambda$ of the form $\bar{P}_\lambda = \{\lambda X + (1-\lambda)F(X)\}, X \in \bar{P}$ with the hyperplane of equation $x_{-1} = 1$.

This implies that, if X_τ belongs to the line in $\mathbb{R}^6_2$ joining the origin and $\tau \in C_P \subset \{x_{-1} = 1\}$, the intersection of the line joining the origin and the point $\lambda X_\tau + (1-\lambda)F(X_\tau)$ with $\{x_{-1} = 1\}$ is contained in the light-ray $\ell_\tau \subset \{x_{-1} = 1\}$. Let us insist: for an arbitrary map $\Psi : C_P \to C_{P^*}$, the points $\lambda\tau + (1-\lambda)\Psi(\tau)$ would have no reason to be contained in the plane P_λ.

Let $\dot{\tau}$ be a vector tangent at τ to the curve C_P, $\dot{\Phi(\tau)}$ a vector tangent at $\Phi(\tau)$ to the curve C_{P^*} and $\dot{\tau_\lambda}$ a vector tangent at $\tau_\lambda = f_1(\lambda,\tau)\tau + f_2(\lambda,\tau)\Phi(\tau)$ to the curve C_{P_λ}. The curve C_{P_λ} can be parameterized by $f_1(\lambda,\tau)\tau + f_2(\lambda,\tau)\Phi(\tau), \tau \in C_P$, therefore $\dot{\tau}_\lambda$ is a linear combination of $\dot{\tau}$, $\dot{\Phi(\tau)}$, τ and $\Phi(\tau)$. This implies that the light-ray ℓ_τ is orthogonal to C_{P_λ} as $\dot{\tau}$ and $\dot{\Phi(\tau)}$, τ and $\Phi(\tau)$ are all orthogonal to ℓ_τ (recall that the light-rays joining the two brother "circles" C_P and C_{P^*} are orthogonal to the tangents to the two brother "circles" at the points where they intersect them, see Proposition 3.4.2).

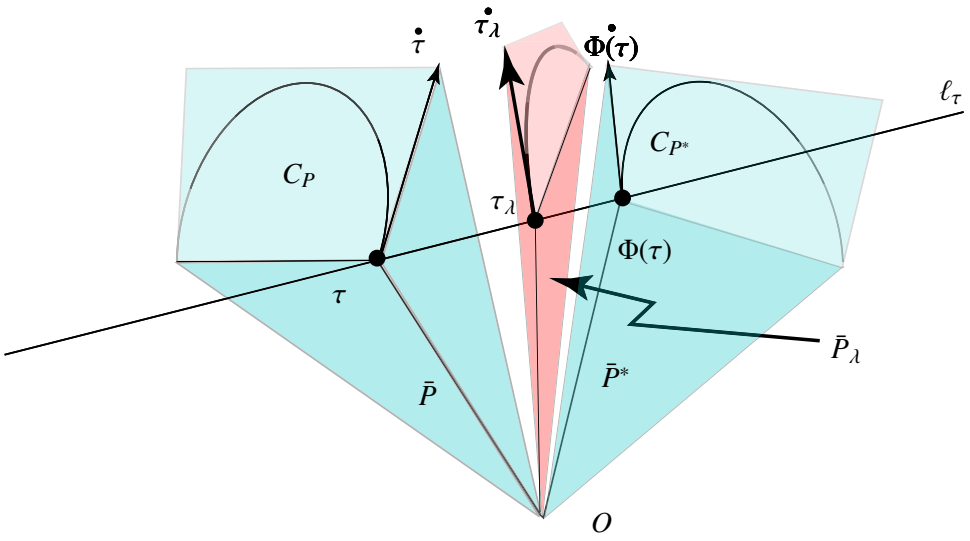


Figure 26: Orthogonal spaces $\bar{P}$, $\bar{P}^*$ in $\mathbb{R}^6_2$ and a subspace $\bar{P}_\lambda \subset \mathbb{R}^6_2$

9 Algorithms

9.1 Determination of a family of Dupin cyclides

With what precedes, it is very easy to understand how to compute the value of the parameters on the light-rays, Formula (7), and Formula (8). Notice that the choice of origin on the light-rays which was convenient for the proofs is not necessary for the algorithm. We can also see that it is not necessary to impose the condition $\mathcal{L}\left(m_i, m_j\right) = -1$, Formula (6).

Remark: With i equals to 1 or 2, let $l_i : t \mapsto \sigma_i + t\, m_i$ be a light-ray belonging to Λ^4. We suppose that these lines are disjoint. From the relation $\mathcal{L}(\sigma_1 + a_1\, m_1, \sigma_2 + b_2\, m_2) = 1$, we deduce that,

The point $l_2(b_2)$ belongs to the tangent hyperplane $T_{l_1(a_1)}\Lambda^4$ if we have $b_2 = f(\sigma_1, m_1, \sigma_2, m_2, a_1)$ where f is given by

$$\begin{array}{rccl} f: & \Lambda^4 \times \mathbb{E}^3 \times \Lambda^4 \times \mathbb{E}^3 \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ & (\sigma_1, m_1, \sigma_2, m_2, a_1) & \longmapsto & \dfrac{1 - \mathcal{L}(\sigma_1, \sigma_2) - a_1 \mathcal{L}(m_1, \sigma_2)}{\mathcal{L}(\sigma_1, m_2) + a_1 \mathcal{L}(m_1, m_2)} \ . \end{array} \tag{13}$$

The same function is called to compute the parameters during the *ping*, *pang* and *pong* steps.

In the algorithm 1, we give a method, when the problem has a one-parameter of solutions, to compute one Dupin cyclide which is tangent to three couples points-planes.

One can note that the condition $a_0 == a_1$ is equivalent to the previous conditions : $\mathcal{A} = 0$ or the composition of the six homographies is the identity.

In the algorithm 1, if we delete the step 3, we obtain a one-parameter family of Dupin cyclides. So, we can compute the centres Ω_a and Ω_b of the two circles passing through the two couples of three points $\left(\sigma_i^a\right)_{[\![1,3]\!]}$ and $\left(\sigma_i^b\right)_{[\![1,3]\!]}$. Each center depends on the parameter a_1 and then, using the sign of $\mathcal{L}(\Omega_a, \Omega_a)$ or $\mathcal{L}(\Omega_b, \Omega_b)$, we can determine the type of the Dupin cyclides :

- if $\mathcal{L}(\Omega_a, \Omega_a) = 0$ or $\mathcal{L}(\Omega_b, \Omega_b) = 0$ then the Dupin cyclide has one singular point ;
- if $\mathcal{L}(\Omega_a, \Omega_a) > 0$ then the Dupin cyclide has two singular points ;
- if $\mathcal{L}(\Omega_a, \Omega_a) < 0$ then the Dupin cyclide has no singular point ;

The figure 27 shows a favorable case : we take three contact conditions on a Dupin cyclide (each tangent plane is oriented by the vector normal to the Dupin cyclide), so we are sure to have a one-parameter family of solutions.

Index	Point M_i	Oriented normal vector $\overrightarrow{n_i}$
$i = 1$	$(7.488, 6.928, 2.044)$	$(-0.2, 0, -0.980)$
$i = 2$	$(-9.551, 6.041, 5.028)$	$(0.171, -0.218, -0.961)$
$i = 3$	$(0.700, -12, 223, 2.425)$	$(-0.2; 0.693, -0.693)$

Table 1: Values of the three $(M_i; \mathcal{P}_i)_{i \in [\![1;3]\!]}$ couples of oriented contact conditions, figure 27. Each plane $\mathcal{P}_i$ is defined by the point M_i and the vector $\overrightarrow{n_i}$.

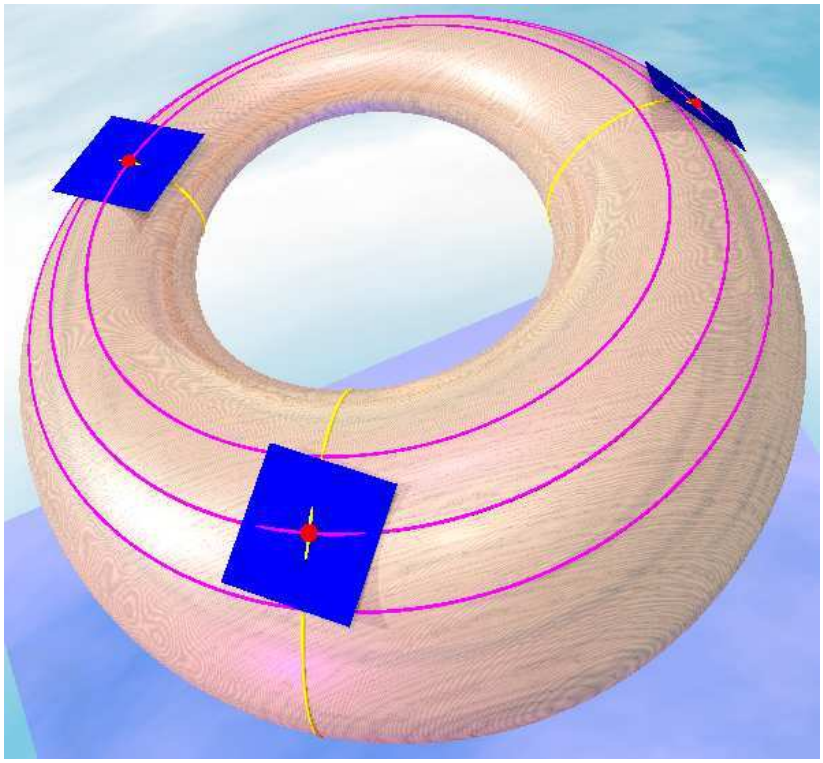


Figure 27: Initialization of the problem with three points belonging to a Dupin cyclide and three planes tangent to the Dupin cyclide at the previous points, table 1.

Figure 28 (resp. Figure 29) shows two examples with ring (resp. horned) Dupin cyclides.

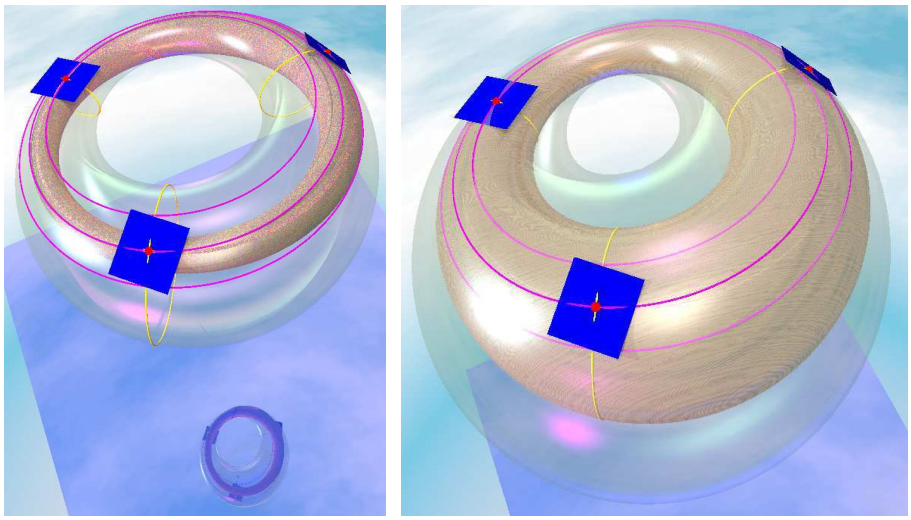
The table 2 shows the parameters of the Dupin cyclides and the affine transformations to put the computed Dupin cyclide in the initial Dupin cyclide orthonormal basis.

Figure 30 shows the common curve : all the Dupin cyclides of a one-parameter family are tangent along this curve.

9.2 Determination of two families of Dupin cyclides

An oriented contact condition can be given by a point and an oriented plane through it, or by a point and an oriented sphere containing the point. Here, as we start with a cyclide C, it is easier to define a contact condition (m, T_mC) by (m, Σ_m) where Σ_m is a sphere tangent at m to C, for example the sphere of one of the two families defining C containing m. In general, if we change the orientation of one of the three spheres, the previous conditions is not satisfied. So, we propose an algorithm which permits the construction of another one-parameter family of Dupin cyclides, algorithm 2.

Figure 31 shows two Dupin cyclides which are tangent only at three points. The green Dupin cyclide is same as in Figure 27. Its parameters are $a = 10, c = 2$ and $\mu = 3.5$. We consider he family of the spheres which are centered on the ellipse. The coordinates of points, centers and the value of the radius of the spheres are given by Table 3.



(a) (b)

Figure 28: Two examples with ring Dupin cyclides (the parameters are given in the table 2). (a) : the texture of the initial Dupin cyclide is glass. (b) : the texture of the initial Dupin cyclide is wood.

The parameters of the other Dupin cyclide are

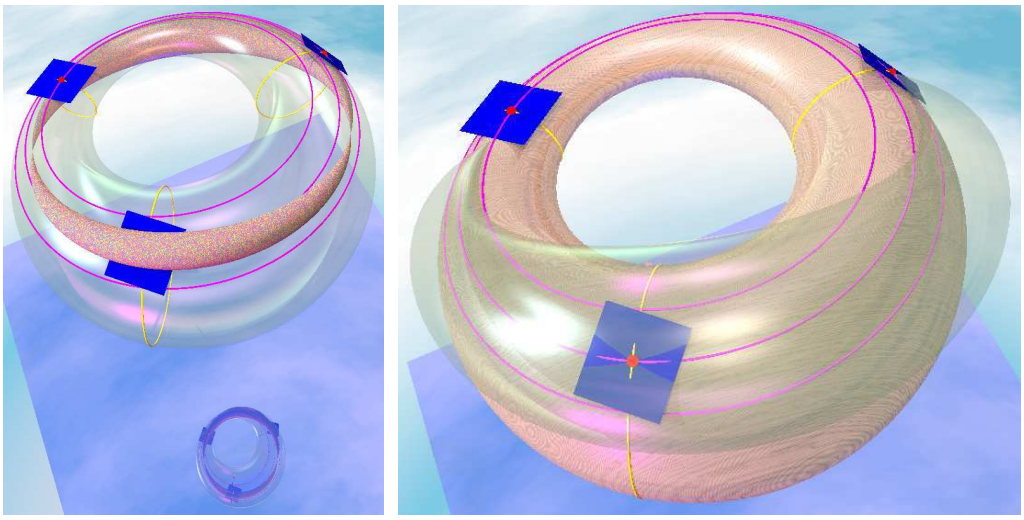
$$(a, c, \mu) \simeq (12.162, 1.300, 1.605)$$

the matrix of transformation is

$$\begin{pmatrix} -0.723 & -0.688 & 0.065 \\ 0.690 & -0.724 & 0.010 \\ 0.040 & 0.052 & 0.998 \end{pmatrix}$$

whereas the translation vector is

$$(-0.330, -1.998, 4.214)$$



(a) (b)

Figure 29: Two examples with horned Dupin cyclides, the texture of the original (ring) Dupin cyclide is glass. (the parameters are given in the table 2)

Figure	(a, c, μ)	Matrix of the transformation	Translation
28(a)	$(10.817, 0.401, 1.316)$	$\begin{pmatrix} -0.254 & -0.949 & 0.188 \\ 0.962 & -0.269 & -0.055 \\ 0.103 & 0.167 & 0.981 \end{pmatrix}$	$(-0.697, -0.303, 2.048)$
28(b)	$(9.658, 4.125, 5.735)$	$\begin{pmatrix} 0.965 & -0.160 & 0.208 \\ 0.142 & -0.985 & -0.098 \\ 0.221 & 0.065 & -0.973 \end{pmatrix}$	$(0.714, 0.310, -2.096)$
29(a)	$(11.365, 1.279, 0.230)$	$\begin{pmatrix} -0.876 & -0.400 & 0.269 \\ 0.396 & -0.915 & -0.073 \\ 0.275 & 0.043 & 0.960 \end{pmatrix}$	$(-1.044, -0.453, 3.066)$
29(b)	$(12.1812, 2.623, 1.1661)$	$\begin{pmatrix} 0.901 & -0.245 & -0.360 \\ -0.229 & -0.960 & 0.087 \\ -0.370 & 0.003 & -0.929 \end{pmatrix}$	$(-1.490, -0.646, 4.374)$

Table 2: Computation of the Dupin cyclides in the orthonormal basis associated to the initial cyclide.

Algorithm 1 Computation of tri-tangent Dupin cyclides.

Input: Three $(M_i; \mathcal{P}_i)_{i\in[\![1;3]\!]}$ couples of oriented contact conditions where each plane is oriented by a normal vector.

1. `For` i `from` 1 `to` 3 `do` computation of m_i, representation of the point M_i in the paraboloid $\mathbb{E}^3$ `od`.
2. `For` i `from` 1 `to` 3 `do` computation of σ_i, representation of the oriented plane $\mathcal{P}_i$ in Λ^4 `od`.
3. Choice of a real number a_1.
4. Computation of $b_2 = f(\sigma_1, m_1, \sigma_2, m_2, a_1)$, (see Formula (13)).
5. Computation of $a_3 = f(\sigma_2, m_2, \sigma_3, m_3, b_2)$.
6. Computation of $b_1 = f(\sigma_3, m_3, \sigma_1, m_1, a_3)$.
7. Computation of $a_2 = f(\sigma_1, m_1, \sigma_2, m_2, b_1)$.
8. Computation of $b_3 = f(\sigma_2, m_2, \sigma_3, m_3, a_2)$.
9. Computation of $a_0 = f(\sigma_3, m_3, \sigma_1, m_1, b_3)$.
10. `if` $a_0 == a_1$

    `then for` i `from` 1 `to` 3 `do`

    Computation of $\sigma_i^a = \sigma_i + a_i\, m_i$.

    Computation of $\sigma_i^b = \sigma_i + b_i\, m_i$.

    `return` two couples of three points $\left(\sigma_i^a\right)_{[\![1,3]\!]}$ and $\left(\sigma_i^b\right)_{[\![1,3]\!]}$ belonging to each brother circle.

    `od`

    `else` there is no solution.

    `fi`

Output: emptyset or a Dupin cyclide.

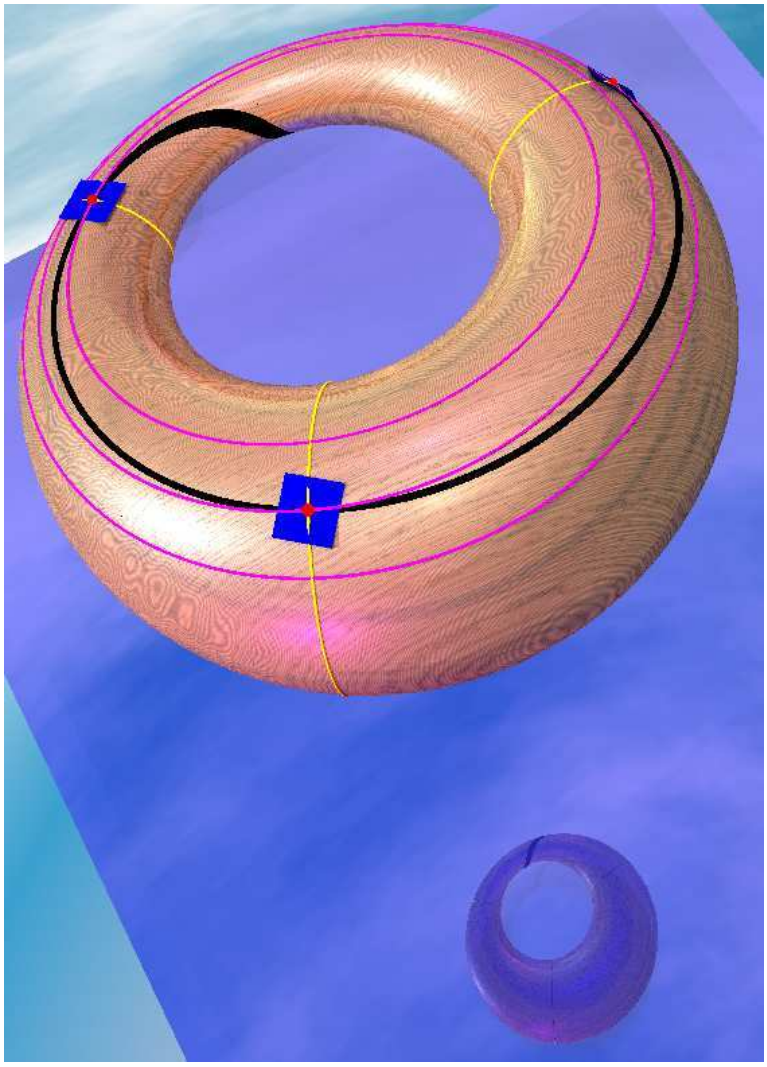


Figure 30: All Dupin cyclides of the family are tangent along a curve.

Index	Point	Dupin cyclide map	Center	Radius
$i = 1$	$\left(\frac{7+48\sqrt{2}}{10}; 4\sqrt{3}; \frac{\sqrt{3}\left(7\sqrt{2}-4\right)}{5}\right)$	$\Gamma\left(\frac{\pi}{4}, -\frac{\pi}{2}\right)$	$\left(5\sqrt{2}, 4\sqrt{3}, 0\right)$	$\frac{7}{2} - \sqrt{2}$
$i = 2$	$\left(\frac{7}{10}; \frac{-\sqrt{3}\left(7+20\sqrt{2}\right)}{5}; \frac{7\sqrt{3}}{5}\right)$	$\Gamma\left(-\frac{\pi}{2}, -\frac{3\pi}{4}\right)$	$\left(0, -4\sqrt{6}, 0\right)$	$\frac{7}{2}$
$i = 3$	$(-9.551, 6.041, 5.027)$	$\Gamma\left(\frac{5\pi}{6}, 4.286\right)$	$\left(-5\sqrt{3}, 2\sqrt{6}, 0\right)$	$\pm\frac{7+\sqrt{3}}{2}$

Table 3: Values of the three points and three spheres (i.e. oriented contact conditions) from the Dupin cyclide map, figure 31.

Algorithm 2 Computation of tri-tangent Dupin cyclides.

Input: A Dupin cyclide C_0 defined by a family of spheres $\Sigma(t)$ with t in $[0, 2\pi]$.

1. Choice of three distinct numbers t_1, t_2 and t_3 in $[0, 2\pi]$.
2. Choice of two points M_1 and M_2 belonging to a characteristic circle on the sphere $\Sigma(t_1)$ and $\Sigma(t_2)$ respectively.
3. Computation of a point $M(\psi)$ belonging to a characteristic circle on the sphere $\Sigma(t_3)$ (the value of ψ is not known).
4. `For` i `from` 1 `to` 2 `do` computation of m_i, representation of the point M_i in the paraboloid $\mathbb{R}^3$ `od`.
5. Computation of $m(\psi)$, representation of the point $M(\psi)$ in the paraboloid $\mathbb{R}^3$.
6. `For` i `from` 1 `to` 3 `do` computation of σ_i, representation of the oriented plane Σ_i in Λ^4 `od`.
7. Choice of a real number a_1.
8. Computation of $b_2 = f(\sigma_1, m_1, \sigma_2, m_2, a_1)$.
9. Computation of $a_3(\psi) = f(\sigma_2, m_2, \sigma_3, m_3(\psi), b_2)$.
10. Computation of $b_1(\psi) = f(\sigma_3, m_3(\psi), \sigma_1, m_1, a_3(\psi))$.
11. Computation of $a_2(\psi) = f(\sigma_1, m_1, \sigma_2, m_2, b_1(\psi))$.
12. Computation of $b_3(\psi) = f(\sigma_2, m_2, \sigma_3, m_3(\psi), a_2(\psi))$.
13. Computation of $a_0(\psi) = f(\sigma_3, m_3(\psi), \sigma_1, m_1, b_3(\psi))$.
14. $E_s = \{\psi \in [0, 2\pi[\mid a_0(\psi) = a_1\}$.
15. `if` $E_s \neq \emptyset$ `then` Choice of a number ψ_3 in E_s.

    Computation of $m_3 = m(\psi_3)$.

    `for` i `from` 1 `to` 3 `do`

    Computation of $\sigma_i^a = \sigma_i + a_i\, m_i$.

    Computation of $\sigma_i^b = \sigma_i + b_i\, m_i$.

    `return` two couples of three points $\left(\sigma_i^a\right)_{[\![1,3]\!]}$ and $\left(\sigma_i^b\right)_{[\![1,3]\!]}$ belonging to each brother circle.

    `od`

    `else` there is no solution.

    `fi`

Output: emptyset or a Dupin cyclide tri-tangent to C_0 only at three points.

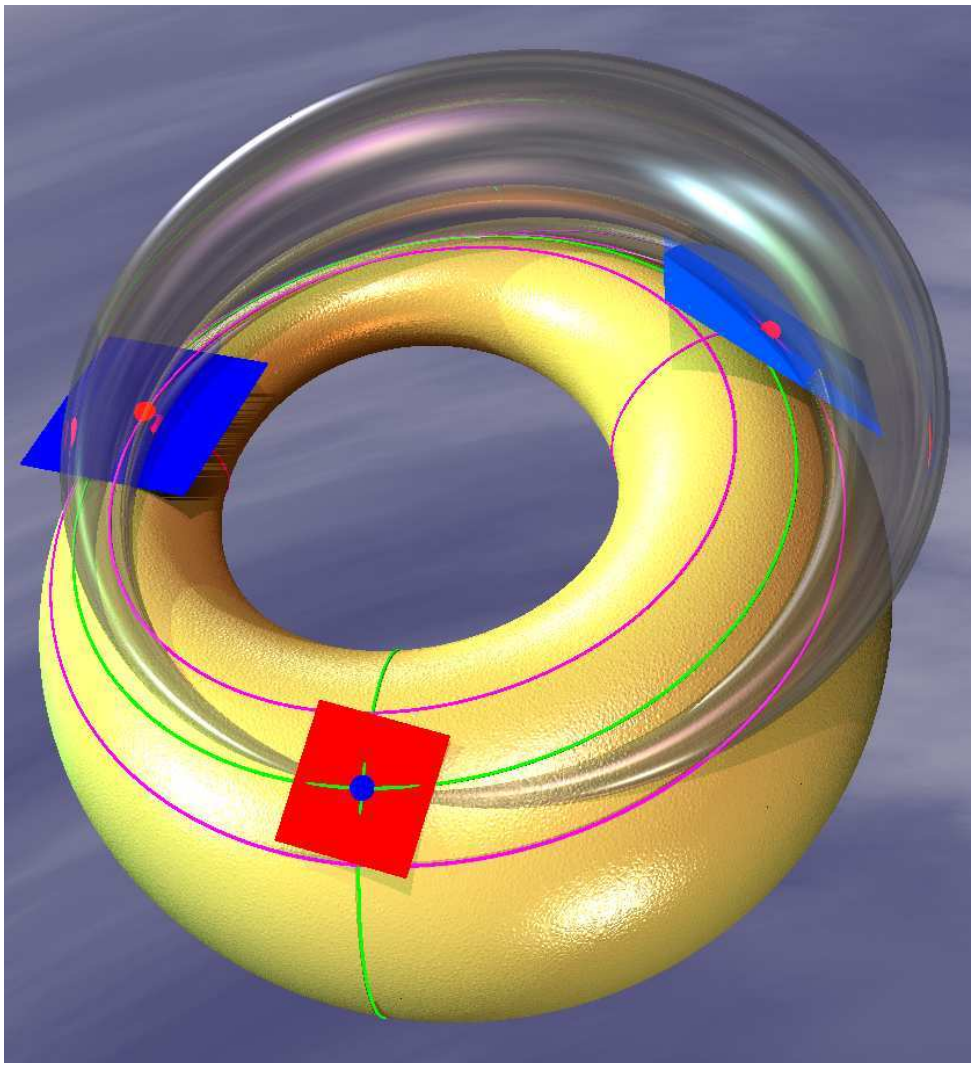


Figure 31: The two Dupin cyclides are tangent only at three points.

10 Conclusion and perspectives

In this paper, when it is possible, we have proposed a method to construct, in the usual 3D Euclidean space, a one-parameter family of Dupin cyclides which are tangent to three given oriented planes (or spheres) at three given points. Sometimes, we can obtain another one-parameter family of Dupin cyclides by changing the orientation of one of the three planes.
To solve the three contacts problem, we have represented the Dupin cyclide in the space of (oriented) spheres, the quadric $\Lambda^4 \subset \mathbb{R}^5_1$. In this space, a Dupin cyclide is defined by two brother circles for the Lorentz form. Our problem is translated into a dynamical problem involving light-rays of Λ^4. The maps between light-rays are homographies. This indicates that projective geometry may also be involved. That is why we construct a 6-dimensional space $\mathbb{R}^6_2$, endowed with a quadratic form of signature two. The Lorentz space $\mathbb{R}^5_1$ is the affine subspace $\mathcal{H}_\mathcal{L}$ of $\mathbb{R}^6_2$ of equation $x_{-1} = 1$. The light-cone of $\mathbb{R}^6_2$ is the cone on $\Lambda^4 \subset \mathcal{H}_\mathcal{L}$.

The linear maps of $\mathbb{R}^6_2$ preserving the cone on Λ^4 plays already an important role in our present work. A deeper understanding will include our work in the frame of Laguerre geometry.

Staying in Λ^4, one can notice that both Dupin Cyclides and quadrics are envelopes of spheres forming a doubly ruled surface, that is a surface filled by two families of geodesics, like the two dimensional hyperboloid of revolution filled by two families of lines. It would by interesting to know if there are envelopes of spheres belonging to a doubly ruled surface other than Dupin cyclides, quadrics and their images by the Möbius group.

We also would like to construct surfaces using 3D triangles made of pieces of cyclides. We hope that these patches will replace the meshes using planar triangles: the computations should be faster because we will need fewer triangles, fewer topological informations to connect the triangles. We can remark that these triangles have a parametric equation and a quartic implicit equation.

Moreover, some particular 3D triangles can have circular edges: the patch is then bounded by two arcs of characteristic circles and one arc of Villarceau circle. We will study the conditions that contact conditions picked on a smooth surface should satisfy in order to obtain such a triangular patch of cyclide.

Another possibility is the computation of subdivisions surfaces which have not the same shape: along a circle arc, it will be possible to join a 3D triangle with a 3D quadrilateral which will be modelized by a rational quadratic Bézier patch representing a Dupin cyclide patch.

Likely to be more difficult is the construction of such a surface made of patches of cyclide from a point cloud. A distance functional from the point cloud to the surface should then be constructed[4].

Finally Dupin cyclides and the *ping* $\circ$ *pong* $\circ$ *pang* map we constructed in this article may become a useful tool to study the local conformal geometry of surfaces in $\mathbb{R}^3$.

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