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# Completely Independent Spanning Trees in Some Regular Networks

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## Abstract

Let  $k \geq 2$  be an integer and  $T_1, \dots, T_k$  be spanning trees of a graph  $G$ . If for any pair of vertices  $(u, v)$  of  $V(G)$ , the paths from  $u$  to  $v$  in each  $T_i$ ,  $1 \leq i \leq k$ , do not contain common edges and common vertices, except the vertices  $u$  and  $v$ , then  $T_1, \dots, T_k$  are completely independent spanning trees in  $G$ . For  $2k$ -regular graphs which are  $2k$ -connected, such as the Cartesian product of a complete graph of order  $2k - 1$  and a cycle and some Cartesian products of three cycles (for  $k = 3$ ), the maximum number of completely independent spanning trees contained in these graphs is determined and it turns out that this maximum is not always  $k$ .

**Keywords:** Spanning tree, Cartesian product, Completely independent spanning tree.

## 1 Introduction

Let  $k \geq 2$  be an integer and  $T_1, \dots, T_k$  be spanning trees in a graph  $G$ . The spanning trees  $T_1, \dots, T_k$  are *edge-disjoint* if  $E(T_1) \cap \dots \cap E(T_k) = \emptyset$ . For a given tree  $T$  and a given pair of vertices  $(u, v)$  of  $T$ , let  $P_T(u, v)$  be the set of vertices in the unique path between  $u$  and  $v$  in  $T$ . The spanning trees  $T_1, \dots, T_k$  are *internally disjoint* if for any pair of vertices  $(u, v)$  of  $V(G)$ ,  $P_{T_1}(u, v) \cap \dots \cap P_{T_k}(u, v) = \{u, v\}$ . Finally, the spanning trees  $T_1, \dots, T_k$  are *completely independent spanning trees* if they are pairwise edge disjoint and internally disjoint.

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Disjoint spanning trees have been extensively studied as they are of practical interest for fault-tolerant broadcasting or load-balancing communication systems in interconnection networks : a spanning-tree is often used in various network operations; computing completely independent spanning-trees guarantees a continuity of service, as each can be immediately used as backup spanning tree if a node or link failure occurs on the current spanning tree. Thus, computing  $k$  completely independent spanning trees allows to handle up to  $k - 1$  simultaneous independent node or link failures. In this context, a network is often modeled by a graph  $G$  in which the set of vertices  $V(G)$  corresponds to the nodes set and the set of edges  $E(G)$  to the set of direct links between nodes.

Completely independent spanning trees were introduced by T. Hasunuma [4] and then have been studied on different classes of graphs, such as underlying graphs of line graphs [4], maximal planar graphs [6], Cartesian product of two cycles [7] and complete graphs, complete bipartite and tripartite graphs [11]. Moreover, the decision problem that consists in determining if there exist two completely independent spanning trees in a graph  $G$  is NP-hard [6].

Other works on disjoint spanning trees include independent spanning trees which focus on finding spanning trees  $T_1, \dots, T_k$  rooted at  $r$ , such that for any vertex  $v$  the paths from  $r$  to  $v$  in  $T_1, \dots, T_k$  are pairwise openly disjoint. the main difference is that  $T_1, \dots, T_k$  are rooted at  $r$  and only the paths to  $r$  are considered. Thus  $T_1, \dots, T_k$  may share common edges, which is not admissible with completely independent spanning trees. Independent spanning trees have been studied in several topologies, including product graphs [10], de Bruijn and Kautz digraphs [3, 5], and chordal rings [9]. Related works also include Edge-disjoint spanning trees, i.e. spanning-trees which are pairwise edge disjoint only. Edge-disjoint spanning trees have been studied on many classes of graphs, including hypercubes [1], Cartesian product of cycles [2] and Cartesian product of two graphs [8].

We use the following notations : for a tree, a vertex that is not a leaf is called an *inner vertex*. For a vertex  $u$  of a graph  $G$ , let  $d_G(u)$  be its degree in  $G$ , i.e. the number of edges of  $G$  incident with it.

For clarity, we recall the definition of the Cartesian product of two graphs : Given two graphs  $G$  and  $H$ , the Cartesian product of  $G$  and  $H$ , denoted  $G \square H$ , is the graph with vertex set  $V(G) \times V(H)$  and edge set  $\{(u, u')(v, v') | (u = v \wedge u'v' \in E(H)) \vee (u' = v' \wedge uv \in E(G))\}$ .

The following theorem gives an alternative definition [4] of completely independent spanning trees.

**Theorem 1.1** ([4]). *Let  $k \geq 2$  be an integer.  $T_1, \dots, T_k$  are completely independent spanning trees in a graph  $G$  if and only if they are edge-disjoint spanning trees of  $G$  and for any  $v \in V(G)$ , there is at most one  $T_i$  such that  $d_{T_i}(v) > 1$ .*

It has been conjectured that in any  $2k$ -connected graph, there are  $k$  completely independent spanning trees [6]. This conjecture has been refuted, as there exist  $2k$ -connected graphs which do not contain two completely independent spanning trees [12], for any integer  $k$ . However, the given counterexamples are not  $2k$ -regular.

**Proposition 1.2** ([12]). *For any  $k \geq 2$ , there exist  $2k$ -connected graphs that do not contain two completely independent spanning trees.*

The proof of the previous proposition consists in constructing a  $2k$ -connected graph with a large proportion of vertices of degree  $2k$  adjacent to the same vertices and proving that these vertices of degree  $2k$  can not be all adjacent to inner vertices in a fixed tree.

This article is organized as follows. Section 2 presents necessary conditions on  $2r$ -regular graphs in order to have  $r$  completely independent spanning trees. Section 3 presents the maximum number of completely independent spanning trees in  $K_m \square C_n$ , for  $n \geq 3$  and  $m \geq 3$ . In particular, we exhibit the first  $2r$ -regular graphs which are  $2r$ -connected and which do not contain  $r$  completely independent spanning trees. In Section 4, we determine three completely independent spanning trees in some Cartesian products of three cycles  $C_{n_1} \square C_{n_2} \square C_{n_3}$ , for  $3 \leq n_1 \leq n_2 \leq n_3$ .

## 2 Necessary conditions on $2r$ -regular graphs

**Proposition 2.1.** *If in a  $2r$ -regular graph  $G$  there exist  $r$  completely independent spanning trees, then every spanning tree has maximum degree at most  $r+1$ .*

*Proof.* By Theorem 1.1, every vertex should be of degree 1 in every spanning tree except in one spanning tree. Hence, in a spanning tree, a vertex is either of degree 1 (a leaf) or has degree between 2 and  $r+1$  (an inner vertex), as  $2r - (r-1) = r+1$ .  $\square$

Let  $IN(T)$  be the set of inner vertices in a tree  $T$ .

**Proposition 2.2.** *If in a  $2r$ -regular graph  $G$  of order  $n$  there exist  $r$  completely independent spanning trees, then there exists a spanning tree  $T$  among them such that  $|IN(T)| \leq \lfloor n/r \rfloor$ .*

*Proof.* Let  $T_1, \dots, T_r$  be completely independent spanning trees in  $G$  and suppose that  $|IN(T_i)| > \lfloor n/r \rfloor$  for every  $i \in \{1, \dots, r\}$ . By Theorem 1.1, we have  $\sum_{i=1}^r |IN(T_i)| \leq n$ . With our hypothesis, we have  $\sum_{i=1}^r |IN(T_i)| \geq (\lfloor n/r \rfloor + 1)r > n$ , and a contradiction.  $\square$

**Proposition 2.3.** *If in a  $2r$ -regular graph  $G$  of order  $n$  there exist  $r$  completely independent spanning trees  $T_1, \dots, T_r$ , then for every integer  $i$ ,  $1 \leq i \leq r$ ,*

$$\left\lceil \frac{n-2}{r} \right\rceil \leq |IN(T_i)| \leq n - \left\lceil \frac{n-2}{r} \right\rceil (r-1).$$

*Proof.* In a spanning tree  $T$  of a graph of order  $n$  we recall that  $\sum_{v \in V(T)} d_T(v) = 2n - 2$ . By Proposition 2.1, we have  $\sum_{v \in V(T)} d_T(v) \leq |IN(T)|r + n$  and we obtain  $\lceil \frac{n-2}{r} \rceil \leq |IN(T)|$ . By Theorem 1.1,  $\sum_{i=1}^r |IN(T_i)| \leq n$ . For a fixed integer  $i$ , using the previous inequality, we obtain  $|IN(T_i)| \leq n - \lceil \frac{n-2}{r} \rceil(r-1)$ .  $\square$

**Definition 2.1.** Let  $G$  be a  $2r$ -regular graph of order  $n$  for which there exist  $r$  completely independent spanning trees  $T_1, \dots, T_r$ . A lost edge is an edge of  $G$  that is in none of the spanning trees  $T_1, \dots, T_r$ . We let  $E^l$  be the set of lost edges, i.e.  $E^l = E(G) - \bigcup_{1 \leq i \leq r} E(T_i)$ . Let also  $E_{T_i}^l = \{uv \in E(G) \mid u, v \in IN(T_i), uv \notin E(T_i)\}$ , for  $i \in \{1, \dots, r\}$ , i.e.  $E_{T_i}^l$  is the subset of edges of  $E^l$  that have their two extremities in  $IN(T_i)$ .

**Proposition 2.4.** If in a  $2r$ -regular graph  $G$  of order  $n$  there exist  $r$  completely independent spanning trees  $T_1, \dots, T_r$ , then  $|E^l| = r$ .

*Proof.* We have  $\sum_{i=1}^r |E(T_i)| + |E^l| = E(G) = rn$  and  $\sum_{i=1}^r |E(T_i)| = r(n-1)$ . Hence,  $|E^l| = r$ .  $\square$

Since each edge of  $E_{T_i}^l$  is also in  $E^l$  and each edge of  $E^l$  is in at most one set  $E_{T_i}^l$  for some integer  $i$ , we have the following observation.

**Observation 2.5.** In a  $2r$ -regular graph  $G$  of order  $n$  for which there exist  $r$  completely independent spanning trees  $T_1, \dots, T_r$ , we have  $\sum_{1 \leq i \leq r} |E_{T_i}^l| \leq |E^l| = r$ .

**Definition 2.2.** The potential extra degree of a spanning tree  $T$  in a  $2r$ -regular graph  $G$  of order  $n$  is  $\text{ped}(T) = |IN(T)|r - n + 2$ .

With Proposition 2.3, we have the following easy observation:

**Observation 2.6.** Let  $G$  be a graph, for which there exist  $r$  completely independent spanning trees  $T_1, \dots, T_r$ . Then, for every  $i$ ,  $0 \leq i \leq r$ ,  $\text{ped}(T_i) \geq 0$ .

Note also that, by definition, the number of inner vertices of  $T_i$  of degree at most  $r$  is bounded by  $\text{ped}(T_i)$ .

**Proposition 2.7.** If in a  $2r$ -regular graph  $G$  of order  $n$  there exist  $r$  completely independent spanning trees, then there exists a spanning tree  $T$  among them such that  $\text{ped}(T) \leq 2$  and  $E_T^l \leq 1$ , with strict inequalities if  $r$  does not divide  $n$ .

*Proof.* By Proposition 2.2, there exists a tree  $T$  among them such that  $|IN(T)| \leq \lfloor n/r \rfloor$ . Hence,  $\text{ped}(T) \leq \lfloor n/r \rfloor r - n + 2 \leq 2$ , with strict inequality if  $r$  does not divide  $n$ . For every edge  $uv$  in  $E_T^l$ , both  $u$  and  $v$  are adjacent to one inner vertex of every spanning tree other than  $T$ . Hence, both  $u$  and  $v$  have degree at most  $r$  in  $T$  and thus  $\text{ped}(T) \geq 2|E_T^l|$ .  $\square$

Note that the inequality  $\text{ped}(T) \geq 2|E_T^l|$  can be strict.

**Corollary 2.8.** *Suppose that  $G$  is a  $2r$ -regular graph of order  $n$  for which there exist  $r$  completely independent spanning trees  $T_1, \dots, T_r$ , for  $r \geq 3$  and  $n \equiv 0 \pmod{r}$ . Then, for every integer  $i$ ,  $1 \leq i \leq r$ ,  $|\text{IN}(T_i)| = n/r$  and  $\text{ped}(T_i) = 2$ .*

**Observation 2.9.** *For a  $2r$ -regular graph  $G$  of order  $n$  for which there exist  $r$  completely independent spanning trees  $T_1, \dots, T_r$ , for every tree  $T_i$ ,  $1 \leq i \leq r$ , and every edge  $e$  in  $E_{T_i}^l$ , the extremities of  $e$  have degree at most  $r$  in  $T_i$ .*

### 3 Cartesian product of a complete graph and a cycle

Let  $m \geq 3$  and  $n \geq 2$  be integers. In this section, the considered graphs are  $K_m \square P_n$ , and  $K_m \square C_n$   $n \geq 3$ .

Let  $V(K_m \square P_n) = V(K_m \square C_n) = \{u_i^j, 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$  and  $E(K_m \square P_n) = \{u_i^j u_k^j, 0 \leq i, k \leq m-1, i \neq k, 0 \leq j \leq n-1\} \cup \{u_i^j u_i^{j+1}, 0 \leq i \leq m-1, 0 \leq j \leq n-2\}$ .  $E(K_m \square C_n) = E(K_m \square P_n) \cup \{u_i^0 u_i^{n-1}, 0 \leq i \leq m-1\}$ .

For  $j \in \{0, \dots, n-1\}$ , the subgraphs  $K^j$  induced by  $\{u_i^j, 0 \leq i \leq m-1\}$  are thus complete graphs on  $m$  vertices that we call  $K$ -copies. In order to study the distribution of inner vertices of the spanning trees among the  $K$ -copies, we let  $V_j(T) = \text{IN}(T) \cap V(K^j)$  and  $n_j(T) = |V_j(T)|$  for any spanning tree  $T$  of  $K_m \square C_n$ .

In the remaining, the subscript of  $u_i^j$  is considered modulo  $m$  and its superscript and the subscripts of  $V_j(T)$  and  $n_j(T)$  are considered modulo  $n$ .

**Proposition 3.1.** *Let  $n$  and  $r$  be integers,  $n \geq 2$ ,  $r \geq 2$ . There exist  $r$  completely independent spanning trees in  $K_{2r} \square P_n$ .*

*Proof.* We construct  $r$  completely independent spanning trees  $T_1, \dots, T_r$  as follows:  $E(T_i) = \{u_{i-1}^j u_{i-1}^{j+1}, u_{r+i-1}^j u_{r+i-1}^{j+1} | j \in \{0, \dots, n-2\}\} \cup \{u_{i-1}^0 u_{r+i-1}^0\} \cup \{u_{i-1}^j u_{i+k}^j, u_{r+i-1}^j u_{r+i+k}^j | k \in \{0, \dots, r-2\}, j \in \{0, \dots, n-1\}\}$ .  $\square$

**Corollary 3.2.** *Let  $n$  and  $r$  be integers,  $n \geq 3$ ,  $r \geq 2$ . There exist  $r$  completely independent spanning trees in  $K_{2r} \square C_n$ .*

In the three next propositions, we will prove that there do not exist  $r$  completely independent spanning trees in  $K_{2r-1} \square C_n$ , for some integers  $r$  and  $n$ . Let  $p = |V(K_{2r-1} \square C_n)| = n(2r-1)$  and assume that there exist  $r$  completely independent spanning trees  $T_1, \dots, T_r$  in  $K_{2r-1} \square C_n$ . Let  $T$  be the spanning tree among them which minimizes  $|\text{IN}(T)|$ , i.e.  $\text{ped}(T)$ . By Proposition 2.2,  $T$  is such that  $|\text{IN}(T)| \leq \lfloor p/r \rfloor = 2n - \lceil n/r \rceil$ ,  $\text{ped}(T) \leq 2nr - \lceil n/r \rceil r - p + 2 \leq n - \lceil n/r \rceil r + 2 \leq 2$  and  $|E_T^l| \leq 1$ . In order to establish this property we will consider all possible distributions of inner vertices of  $T$  among the different  $K$ -copies and prove that for each of them we have a contradiction.

The properties given in the following lemma will be useful.

**Lemma 3.3.** *Let  $a_i(T)$  be the number of  $K$ -copies which contains exactly  $i$  inner vertices of  $T$ . The distribution of inner vertices among the different  $K$ -copies is such that:*

- i) *if  $n_j(T) \geq k$ , for some integer  $j$ , then  $|E_T^l| \geq \frac{1}{2}(k-1)(k-2)$ ;*
- ii)  *$n_j(T) < 4$ , for every integer  $j$ ;*
- iii)  *$a_3(T) \leq 1$ ;*
- iv) *if  $a_3(T) = 1$ , then  $n \equiv 0 \pmod{r}$  and  $n \geq r$ ;*
- v) *if  $a_0(T) = 0$ , then  $a_3(T) \leq a_1(T) - \lceil n/r \rceil$ ; in particular  $a_1(T) > a_3(T)$  and  $a_1(T) \geq \lceil n/r \rceil$ .*

*Proof.* i) : A complete graph of order  $k$  contains  $\frac{1}{2}k(k-1)$  edges and only  $k-1$  edges are in  $E(T)$ . Thus we have  $|E_T^l| \geq \frac{1}{2}(k-1)(k-2)$ .

ii) and iii) : If  $n_j(T) \geq 4$  for some  $j$  or  $a_3(T) > 1$ , then by i), we have  $|E_T^l| \geq 2$ . Hence, a contradiction.

iv) : As  $\text{ped}(T) \leq n - \lceil n/r \rceil r + 2$ , we have  $|E_T^l| < 1$  in the case  $n \not\equiv 0 \pmod{r}$ . As  $n > 0$ , we have  $n \geq r$ .

v) : By ii), we have  $|\text{IN}(T)| = a_1(T) + 2a_2(T) + 3a_3(T)$  and  $a_2 = n - a_1(T) - a_3(T)$ . Hence  $|\text{IN}(T)| = a_1(T) + 2(n - a_1(T) - a_3(T)) + 3a_3(T) \leq 2n - \lceil n/r \rceil$  by the choice of  $T$ . Thus,  $a_3(T) \leq a_1(T) - \lceil n/r \rceil$  and consequently  $a_1(T) > a_3$  and  $a_1(T) \geq \lceil n/r \rceil$ .  $\square$

We recall the following observation used in [12].

**Observation 3.4** ([12]). *If in a graph  $G$  there exist  $r$  completely independent spanning trees  $T_1, \dots, T_r$ , then for every integer  $i$ ,  $1 \leq i \leq r$ , every vertex is adjacent to an inner vertex of  $T_i$ .*

**Proposition 3.5.** *Let  $n, r$  be integers, with  $n \geq 3$  and  $r \geq 6$ . There do not exist  $r$  completely independent spanning trees in  $K_{2r-1} \square C_n$ .*

*Proof.* The proof is by contradiction, using Properties i)-v) of Lemma 3.3. Suppose that there exist  $r$  completely independent spanning trees in  $K_{2r-1} \square C_n$  and let  $T$  be the tree from Proposition 2.2. If a  $K$ -copy  $K^i$ ,  $1 \leq i \leq n$ , contains no inner vertex, then, by Observation 3.4,  $n_{i-1}(T) + n_{i+1}(T) \geq 2r - 1 \geq 11$ . Consequently, we have  $n_{i-1}(T) \geq 6$  or  $n_{i+1}(T) \geq 6$ , contradicting Property ii). Hence  $a_0(T) = 0$ .

By Property v),  $a_1(T) \geq \lceil n/r \rceil \geq 1$ . Hence there exists an integer  $i$ ,  $0 \leq i \leq n-1$ , such that  $n_i = 1$ . Let  $u$  be the (unique) vertex of  $V_i(T)$ . The vertex  $u$  has degree at most  $r+1$  in  $T$  and is adjacent in  $T$  to a vertex of  $V_{i-1}(T) \cup V_{i+1}(T)$ . Then,  $u$  is adjacent in  $T$  to at most  $r$  vertices of  $V(K^i)$ . Thus, at least  $r-2 \geq 4$  vertices are not adjacent in  $T$  to  $u$ . Hence, these  $r-2$  vertices are adjacent in  $T$  to vertices of  $V_{i-1}(T) \cup V_{i+1}(T)$  and consequently  $n_{i-1}(T) + n_{i+1}(T) \geq 5$ . Therefore, we have  $n_{i-1}(T) \geq 3$  or  $n_{i+1}(T) \geq 3$ .

Assume, without loss of generality, that  $n_{i+1}(T) \geq 3$ . By Property ii),  $n_{i+1}(T) = 3$  and by Property iii),  $a_3(T) = 1$ , i.e.,  $n_j(T) < 3$  for any  $j \neq i$ .

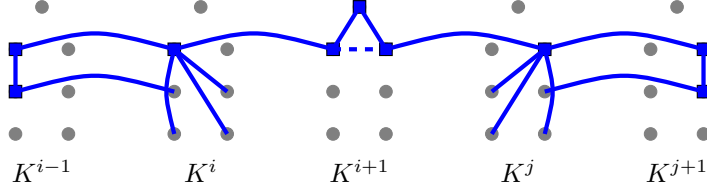


Figure 1: A configuration of inner vertices in the proof of Proposition 3.6. Boxes are inner vertices and the dashed edge represents a lost edge.

But, by Property iv),  $n \geq r$  and by Property v),  $a_1 \geq 2$ . Let  $j$  be such that  $n_j(T) = 1$ , with  $j \neq i$ . Using a similar argument than above, we obtain that  $n_{j-1}(T) \geq 3$  or  $n_{j+1}(T) \geq 3$ . But, as  $a_3(T) = 1$ , the only possibility is to have  $j = i + 2$ , i.e. both  $K$ -copies with one internal vertices are adjacent to the same  $K$ -copy with three internal vertices.

Let  $v$  be the (unique) vertex of  $V_j(T)$ . One vertex among  $u$  and  $v$  is adjacent in  $T$  to two inner vertices (if not  $T$  would be not connected). Suppose, without loss of generality, that  $u$  is adjacent in  $T$  to two inner vertices. Then  $u$  is adjacent in  $T$  to at most  $r - 1$  vertices in  $V(K^i)$ . Thus, at least  $r - 1 \geq 5$  vertices are not adjacent in  $T$  to  $u$ . Therefore, at least 5 vertices are adjacent in  $T$  to vertices of  $V_{i-1}(T) \cup V_{i+1}(T)$  and consequently  $n_{i-1}(T) + n_{i+1}(T) \geq 7$ . Hence, we have  $n_{i-1}(T) \geq 4$  or  $n_{i+1}(T) \geq 4$ , contradicting Property ii).  $\square$

**Proposition 3.6.** *Let  $n, r$  be integers, with  $4 \leq r \leq 5$  and  $n \geq r + 1$ . There do not exist  $r$  completely independent spanning trees in  $K_{2r-1} \square C_n$ .*

*Proof.* The proof is by contradiction, using Properties i)-v) of Lemma 3.3. Suppose that there exist  $r$  completely independent spanning trees in  $K_{2r-1} \square C_n$  and let  $T$  be the tree from Proposition 2.2. If a  $K$ -copy  $K^i$ ,  $0 \leq i \leq n - 1$ , contains no inner vertex, then  $n_{i-1}(T) + n_{i+1}(T) \geq 7$ . Consequently, we have  $n_{i-1}(T) \geq 4$  or  $n_{i+1}(T) \geq 4$ , contradicting Property ii). Hence  $a_0(T) = 0$ . By Property v),  $a_1(T) \geq \lceil n/r \rceil \geq 2$ . Thus, there exist two integers  $i$  and  $j$ ,  $0 \leq i \leq j \leq n - 1$ , such that  $n_i(T) = n_j(T) = 1$ , with  $u \in V_i(T)$  and  $v \in V_j(T)$ .

First, suppose that  $i = j - 1$ . Each of  $u$  and  $v$  has degree at most  $r + 1$  in  $T$  and  $u$  ( $v$ , respectively) is adjacent in  $T$  to a vertex of  $V_{i-1}(T) \cup V_{i+1}(T)$  (of  $V_{j-1}(T) \cup V_{j+1}(T)$ , respectively).

If  $u$  and  $v$  are adjacent in  $T$ , then one vertex among  $u$  and  $v$  is adjacent in  $T$  to a vertex of  $V_{i-1}(T) \cup V_{j+1}(T)$  (if not  $T$  would be not connected). Suppose, without loss of generality, that  $u$  is adjacent to two inner vertices. Then, at least  $r - 1 \geq 3$  vertices of  $V(K^i)$  are not adjacent in  $T$  to  $u$ . Consequently,  $n_{i-1}(T) \geq 4$  and we have a contradiction with Property ii).

Else if  $u$  and  $v$  are not adjacent in  $T$ , then both  $u$  and  $v$  are adjacent in  $T$  to vertices of  $V_{i-1}(T) \cup V_{j+1}(T)$  (if not,  $T$  would be not connected). The vertices  $u$  and  $v$  are each adjacent in  $T$  to at most  $r$  vertices in  $V(K^i) \cup V(K^j)$ . Hence,



there remain at least  $4r - 2 - 2r - 2 = 2r - 4 \geq 4$  vertices in  $V(K^i) \cup V(K^j)$  that must be adjacent in  $T$  to vertices of  $V_{i-1}(T) \cup V_{j+1}(T)$  other than the neighbors of  $u$  and of  $v$ . Consequently  $n_{i-1}(T) + n_{j+1}(T) \geq 6$ . Hence, we have  $n_{i-1}(T) \geq 3$  and  $n_{j+1}(T) \geq 3$ , contradicting Property iii) or  $n_{i-1}(T) \geq 4$  or  $n_{j+1}(T) \geq 4$ , contradicting Property ii).

Second, if  $|i - j| > 1$ , then one vertex among  $u$  and  $v$  is adjacent in  $T$  to two inner vertices (if not  $T$  would be not connected). Suppose, without loss of generality, that  $u$  is adjacent to two inner vertices. At least  $r - 1$  vertices of  $V(K^i)$  are not adjacent in  $T$  to  $u$ . Hence, if  $r = 5$ , we have  $n_{i-1}(T) \geq 3$  and  $n_{i+1}(T) \geq 3$ , contradicting Property iii) or  $n_{i-1}(T) \geq 4$  or  $n_{i+1}(T) \geq 4$ , contradicting Property ii). Consequently, we suppose that  $r = 4$ . Then, at least  $r - 1 \geq 3$  vertices of  $V(K^i)$  are not adjacent in  $T$  to  $u$ . Therefore, we have  $n_{i-1}(T) \geq 3$  or  $n_{i+1}(T) \geq 3$ .

Assume, without loss of generality, that  $n_{i+1}(T) \geq 3$ . By Property ii),  $n_{i+1}(T) = 3$  and by Property iii),  $a_3(T) = 1$ , i.e.,  $n_j(T) < 3$  for any  $j \neq i$ . But, as  $n > r$  and by Property v),  $a_1 \geq 3$ . Let  $i'$  be such that  $n_{i'}(T) = 1$ , with  $i' \neq i$  and  $i' \neq i$ . If  $|i' - i| = 1$  or  $|i' - j| = 1$ , we have a contradiction, using the first point. Two vertices among  $u$ ,  $v$  and  $u'$  should be adjacent to two inner vertices. Suppose it is the vertices  $u$  and  $v$ . Using a similar argument than above, we obtain that  $n_{j-1}(T) \geq 3$  or  $n_{j+1}(T) \geq 3$ . But, as  $a_3(T) = 1$ , the only possibility is to have  $j = i + 2$ , i.e. both  $K$ -copies with one internal vertices are adjacent to the same  $K$ -copy with three internal vertices.

In this case, as  $r = 4$ , then four vertices are not inner vertices in  $V(K^{i+1})$ , at least three vertices of  $V(K^i)$  are not adjacent in  $T$  to  $u$  and at least three vertices of  $V(K^j)$  are not adjacent in  $T$  to  $v$ . Moreover, we have  $n_{i-1}(T) \leq 2$  and  $n_{j+1}(T) \leq 2$ . Figure 1 illustrates this configuration. Thus, four vertices of  $V(K^{i+1})$  are adjacent in  $T$  to vertices of  $V_{i+1}(T)$  and four vertices of  $V(K^i) \cup V(K^j)$  are adjacent in  $T$  to vertices of  $V_{i+1}(T)$ . However, by Observation 2.9, the vertices of  $V_{i+1}(T)$  can be adjacent to at most seven leaves in  $T$ . Hence, we have a contradiction.  $\square$

**Proposition 3.7.** *There do not exist five completely independent spanning trees in  $K_9 \square C_3$ .*

*Proof.* Suppose that there exist five completely independent spanning trees in  $K_9 \square C_3$  and let  $T$  be the tree from Proposition 2.2. We recall that  $|V(K_9 \square C_3)| = 27$  and  $|\text{IN}(T)| \leq 6 - \lceil 3/4 \rceil = 5$ . If a  $K$ -copy  $K^i$ ,  $0 \leq i \leq n - 1$ , contains no inner vertex, then  $n_{i-1}(T) \geq 5$  or  $n_{i+1}(T) \geq 5$ . Thus, we have a contradiction with Property ii). By property iv), as  $n \not\equiv 0 \pmod{r}$ , we have  $a_3(T) = 0$ . Thus, the only possible distribution of inner vertices of  $T$  is  $a_1(T) = 1$  and  $a_2(T) = 2$ . Without loss of generality, suppose that  $n_0(T) = 1$ ,  $n_1(T) = 2$  and  $n_2(T) = 2$ , with  $u \in V_1(T)$ .

Let the position of a vertex  $u_i^j$  be  $i$ . As  $T$  should be connected, two pairs of inner vertices in different  $K$ -copies should be adjacent in  $T$  among these five inner vertices. Thus, these five vertices have only three different positions. The

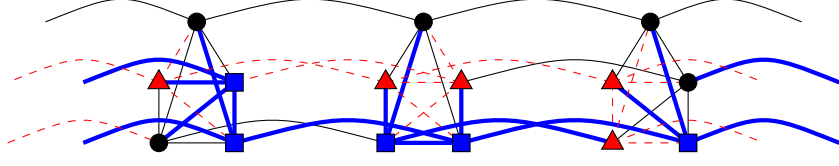


Figure 2: A pattern to have three completely independent spanning trees in  $K_5 \square C_n$ , for  $n \equiv 0 \pmod{3}$ .

vertex  $u$  has degree at most 6 in  $T$ . Hence, there are  $r - 2 \geq 3$  vertices of  $V(K^1)$  not adjacent in  $T$  to  $u$ . As the inner vertices have only two positions different from the position of  $u$ , it is impossible that every vertex is adjacent in  $T$  to an inner vertex of  $T$ .  $\square$

We now show positive results for the remaining values of  $r$  and  $n$ . Some of the spanning trees were found using a computer to solve an ILP formulation of the problem.

**Proposition 3.8.** *Let  $n \geq 3$  be an integer such that  $n \equiv 0 \pmod{3}$ . There exist three completely independent spanning trees in  $K_5 \square C_n$ .*

*Proof.* We construct three completely independent spanning trees  $T_1$ ,  $T_2$  and  $T_3$  using repeatedly the pattern illustrated in Figure 2 on each three consecutive  $K$ -copies:

$$\begin{aligned}
 E(T_1) &= \{u_0^{3j} u_0^{1+3j}, u_0^{1+3j} u_0^{2+3j}, u_0^{2+3j} u_0^{3+3j}, u_0^{3j} u_2^{3j}, u_0^{3j} u_3^{3j}, \\
 &\quad u_3^{3j} u_1^{3j}, u_3^{3j} u_4^{3j}, u_3^{3j} u_3^{1+3j}, u_0^{1+3j} u_1^{1+3j}, u_0^{1+3j} u_4^{1+3j}, u_2^{1+3j} u_2^{2+3j}, \\
 &\quad u_0^{2+3j} u_2^{2+3j}, u_0^{2+3j} u_1^{2+3j}, u_2^{2+3j} u_3^{2+3j}, u_2^{2+3j} u_4^{2+3j} | j \in \{0, \dots, n/3 - 1\}\} - \{u_0^0, u_0^1\}; \\
 E(T_2) &= \{u_1^{3j} u_1^{1+3j}, u_1^{1+3j} u_1^{2+3j}, u_1^{2+3j} u_1^{3+3j}, u_1^{3j} u_0^{3j}, u_1^{3j} u_4^{3j}, \\
 &\quad u_2^{3j} u_2^{1+3j}, u_1^{1+3j} u_2^{1+3j}, u_1^{1+3j} u_4^{1+3j}, u_2^{1+3j} u_0^{1+3j}, u_2^{1+3j} u_3^{1+3j}, u_1^{2+3j} u_3^{2+3j}, \\
 &\quad u_1^{2+3j} u_2^{2+3j}, u_3^{2+3j} u_0^{2+3j}, u_3^{2+3j} u_4^{2+3j}, u_3^{2+3j} u_3^{3+3j} | j \in \{0, \dots, n/3 - 1\}\} - \{u_1^0, u_1^1\}; \\
 E(T_3) &= \{u_4^{3j} u_4^{1+3j}, u_4^{1+3j} u_4^{2+3j}, u_4^{2+3j} u_4^{3+3j}, u_2^{3j} u_4^{3j}, u_2^{3j} u_1^{3j}, \\
 &\quad u_2^{3j} u_3^{3j}, u_4^{3j} u_0^{3j}, u_3^{1+3j} u_4^{1+3j}, u_3^{1+3j} u_0^{1+3j}, u_3^{1+3j} u_1^{1+3j}, u_4^{1+3j} u_2^{1+3j}, \\
 &\quad u_3^{1+3j} u_3^{2+3j}, u_4^{2+3j} u_0^{2+3j}, u_4^{2+3j} u_1^{2+3j}, u_2^{2+3j} u_2^{3+3j} | j \in \{0, \dots, n/3 - 1\}\} - \{u_4^0, u_4^1\}.
 \end{aligned}$$

**Proposition 3.9.** *Let  $n \geq 3$  be an integer. There exist three completely independent spanning trees in  $K_5 \square C_n$ .*

*Proof.* By Proposition 3.8, there exist three completely independent spanning trees in  $K_5 \square C_n$ , for  $n \equiv 0 \pmod{3}$ . For  $n \equiv 1 \pmod{3}$ , we use the pattern from Proposition 3.8 for  $K^4 \cup \dots \cup K^{n-1}$ , completed by the pieces of three completely independent spanning trees of  $K^0 \cup K^1 \cup K^2 \cup K^3$  depicted in Figure 3 and whose edge sets are given in Appendix A.1. For  $n \equiv 2 \pmod{3}$ , we use the pattern from Proposition 3.8 for  $K^5 \cup \dots \cup K^{n-1}$ , completed by the pieces of three

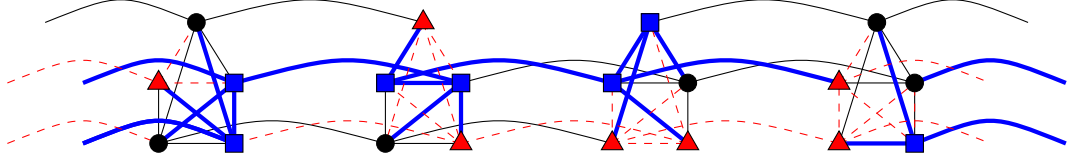


Figure 3: The three completely independent spanning trees in  $K_5 \square C_n$ , for  $K^0 \cup K^1 \cup K^2 \cup K^3$  and  $n \equiv 1 \pmod{3}$ .

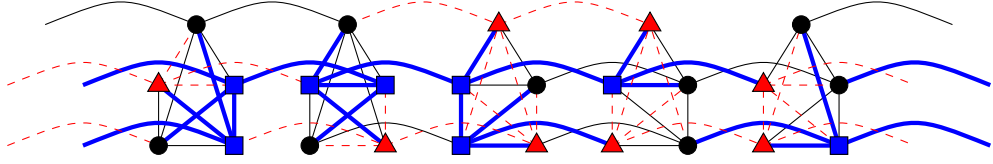


Figure 4: The three completely independent spanning trees in  $K_5 \square C_n$ , for  $K^0 \cup K^1 \cup K^2 \cup K^3 \cup K^4$  and for  $n \equiv 2 \pmod{3}$ .

completely independent spanning trees of  $K^0 \cup K^1 \cup K^2 \cup K^3 \cup K^4$  depicted in Figure 4 and whose edge sets are given in Appendix A.2. Note that Figures 3 and 4 depict also three completely independent spanning trees in  $K_5 \square C_4$  and  $K_5 \square C_5$ .  $\square$

**Proposition 3.10.** *There exist four completely independent spanning trees in  $K_7 \square C_3$ .*

*Proof.* The four completely independent spanning trees in  $K_7 \square C_3$  are depicted in Figure 5 and their edge sets are given in Appendix A.3.  $\square$

**Proposition 3.11.** *There exist four completely independent spanning trees in  $K_7 \square C_4$ .*

*Proof.* The four completely independent spanning trees in  $K_7 \square C_4$  are depicted in Figure 6 and their edge sets are given in Appendix A.4.  $\square$

**Proposition 3.12.** *There exist five completely independent spanning trees in  $K_9 \square C_4$ .*

*Proof.* The five completely independent spanning trees in  $K_9 \square C_4$  are depicted in Figure 7 and their edge sets are given in Appendix A.5.  $\square$

**Proposition 3.13.** *There exist five completely independent spanning trees in  $K_9 \square C_5$ .*

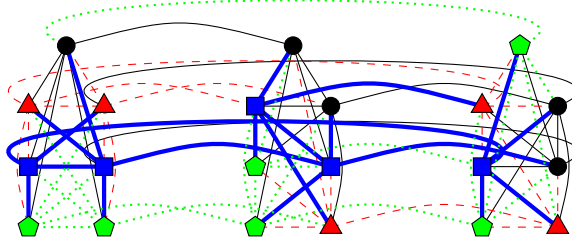


Figure 5: Four completely independent spanning trees in  $K_7 \square C_3$ .

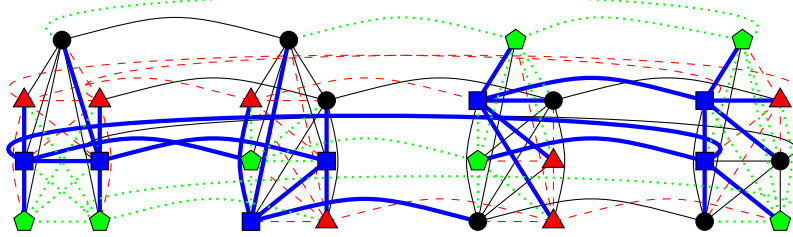


Figure 6: Four completely independent spanning trees in  $K_7 \square C_4$ .

*Proof.* The five completely independent spanning trees in  $K_9 \square C_5$  are depicted in Figure 8 and their edge sets are given in Appendix A.6.  $\square$

We end this section with a theorem summarizing the results for  $K_m \square C_n$ . Given a graph  $G$ , let  $\text{mcist}(G)$  be the maximum integer  $k$  such that there exist  $k$  completely independent spanning trees in  $G$ .

**Theorem 3.14.** *Let  $m \geq 3$  and  $n \geq 3$  be integers. We have:*  

$$\text{mcist}(K_m \square C_n) = \begin{cases} \lceil m/2 \rceil, & \text{if } (m = 3, 5 \vee (m = 7 \wedge n = 3, 4) \vee (m = 9 \wedge n = 4, 5)); \\ \lfloor m/2 \rfloor, & \text{otherwise.} \end{cases}$$

*Proof.* For every even  $m$ , by Corollary 3.2, there exist  $m/2$  completely independent spanning trees. Suppose  $m$  is odd. For  $m = 3$ , Hasunuma and Morisaka [7] has proven that in any Cartesian product of 2-connected graphs, there are two completely independent spanning trees. By Propositions 3.12, 3.13, 3.10, 3.11 and 3.9, we obtain that there exist  $\lceil m/2 \rceil$  completely independent spanning trees for  $m = 5$  or  $(m = 7 \wedge n = 3, 4)$  or  $(m = 9 \wedge n = 4, 5)$ .

In the other cases, by Propositions 3.5, 3.6, 3.7, there do not exist  $\lceil m/2 \rceil$  completely independent spanning trees in these graphs. By Corollary 3.2, there exist  $\lfloor m/2 \rfloor$  completely independent spanning trees in  $K_{m-1} \square C_n$ . From these  $\lfloor m/2 \rfloor$  completely independent spanning trees in  $K_{m-1} \square C_n$ , we can construct  $\lfloor m/2 \rfloor$  completely independent spanning trees in  $K_m \square C_n$ . The graph  $K_m \square C_n$  contains  $n$  vertices  $u_0, \dots, u_{n-1}$  not in  $K_{m-1} \square C_n$ , with  $u_j \in V(K^j)$  for  $j =$

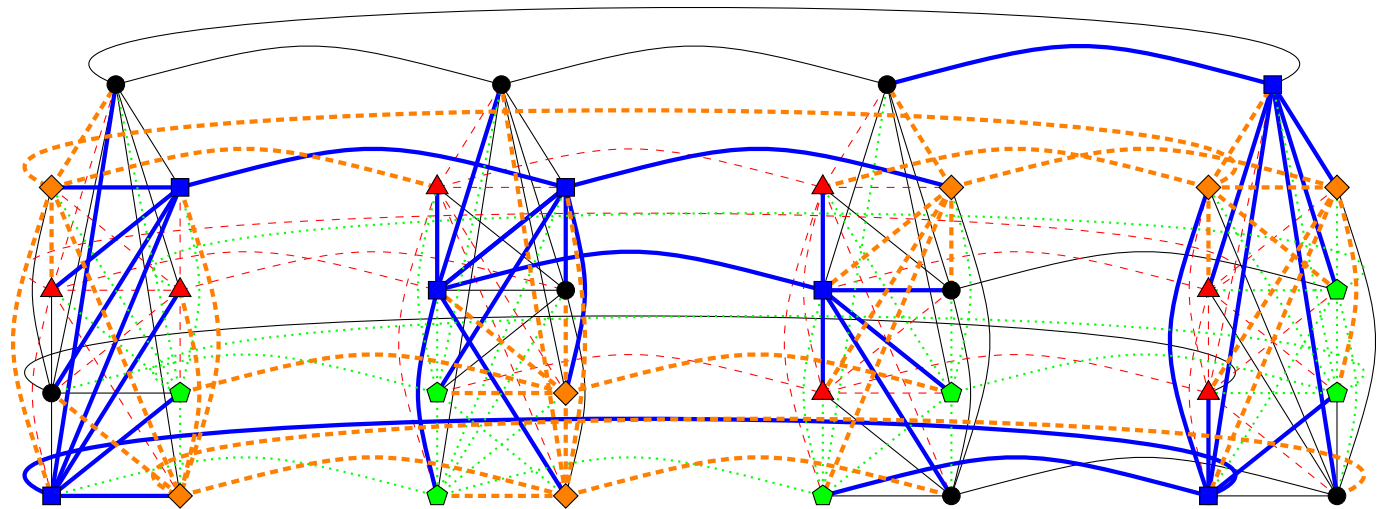


Figure 7: Five completely independent spanning trees in  $K_9 \square C_4$ .

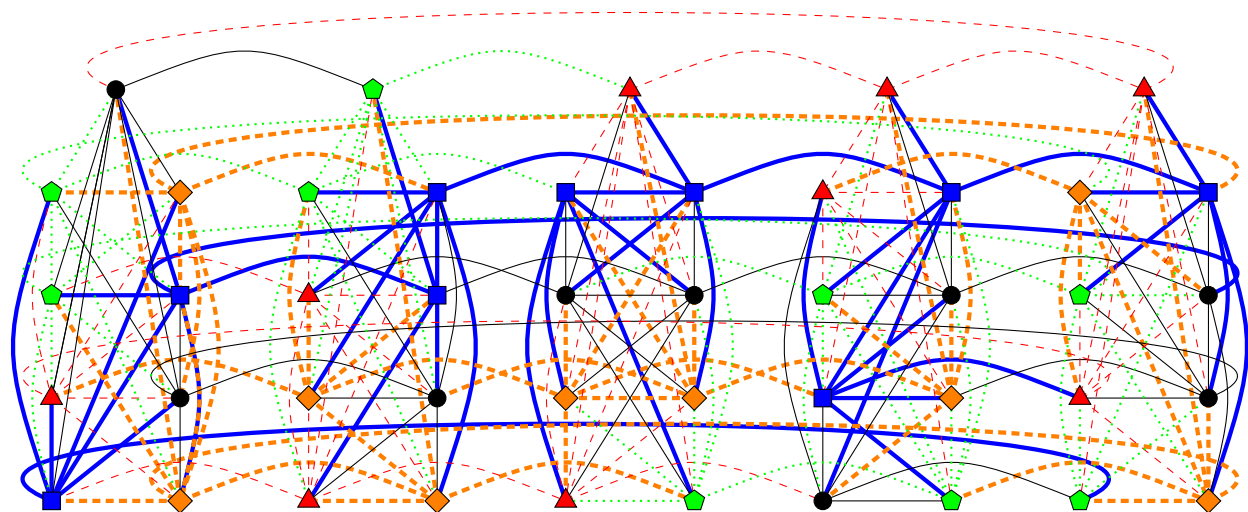


Figure 8: Five completely independent spanning trees in  $K_9 \square C_5$ .

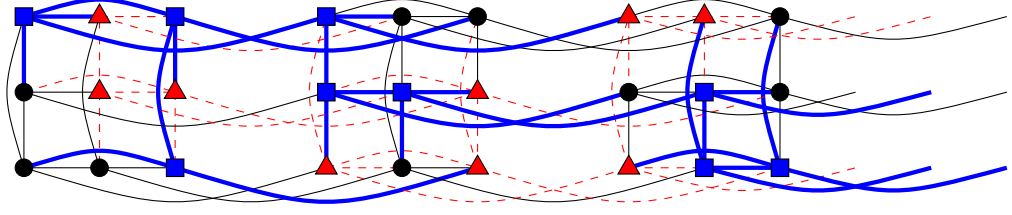


Figure 9: The pattern for the three completely independent spanning trees of  $TM(3, 3, 3q)$ , with  $q \geq 2$ .

$0, \dots, n-1$ . For each  $1 \leq i \leq \lfloor m/2 \rfloor$ , it suffices to add an edge between  $u_j$ ,  $1 \leq j \leq n$ , and a vertex of  $V_j(T_i)$  to obtain  $\lfloor m/2 \rfloor$  completely independent spanning trees in  $K_m \square C_n$ .  $\square$

## 4 3-dimensional toroidal grids

Hasunuma and Morisaka [7] have shown that there are two completely independent spanning trees in any 2-dimensional toroidal grid and left as an open problem the question of whether there are  $n$  completely independent spanning trees in any  $n$ -dimensional toroidal grid, for  $n \geq 3$ . In this section we give a partial answer for  $n = 3$  by finding three completely independent spanning trees in some 3-dimensional toroidal grids.

Let  $n_1, n_2$  and  $n_3$  be positive integers,  $3 \leq n_1 \leq n_2 \leq n_3$ . The 3-dimensional toroidal grid  $TM(n_1, n_2, n_3)$  is the Cartesian product of three cycles:  $C_{n_1} \square C_{n_2} \square C_{n_3}$ . We let  $V(TM(n_1, n_2, n_3)) = \{(i, j, k) | 0 \leq i < n_1, 0 \leq j < n_2, 0 \leq k < n_3\}$  and  $E(TM(n_1, n_2, n_3)) = \{(i, j, k) (i', j', k') | i \equiv i' \pm 1 \pmod{n_1}, j = j', k = k' \vee i = i', j \equiv j' \pm 1 \pmod{n_2}, k = k' \vee i = i', j = j', k \equiv k' \pm 1 \pmod{n_3}\}$ . In the remainder of the section, the integers  $i, j$  and  $k$  in a vertex  $(i, j, k)$  are considered modulo  $n_1, n_2$  and  $n_3$ , respectively.

By a *level* of  $TM(3, 3, q)$  we mean a subgraph of it induced by the vertices with the same third coordinate.

**Proposition 4.1.** *Let  $p, p'$  and  $q$  be positive integers such that  $\gcd(p, p', q) = 1$ . There exist three completely independent spanning trees in  $TM(3p, 3p', 3q)$ .*

*Proof.* We define three completely independent spanning trees  $T_1, T_2$  and  $T_3$  in  $TM(3p, 3p', 3q)$  as follows: for  $j \in \{0, 1, 2\}$ ,

$$E(T_{j-1}) = \{(i+j, j-i, i)(1+i+j, -i+j, i), (i+j, j-i, i)(i+j, 1-i+j, i), (i+j, j-i, i)(i+j, j-i, 1+i), (i+j, j-i, i)(i+j, -1-i+j, i), (1+i+j, j-i, i)(2+i+j, j-i, i), (1+i+j, j-i, i)(1+i+j, j-i, 1+i), (1+i+j, j-i, i)(1+i, j-i-1, i), (i+j, 1-i+j, i)(1+i+j, 1-i+j, i), (i+j, 1-i+j, i)(i+j, 1-i+j, 1+i) | i \in \{0, \dots, pp'q-1\} - (j, j+1, 0)(j, j+1, -1)\}.$$

We require  $\gcd(p, p', q) = 1$ , in order that  $T_1, T_2, T_3$  contain every vertex of

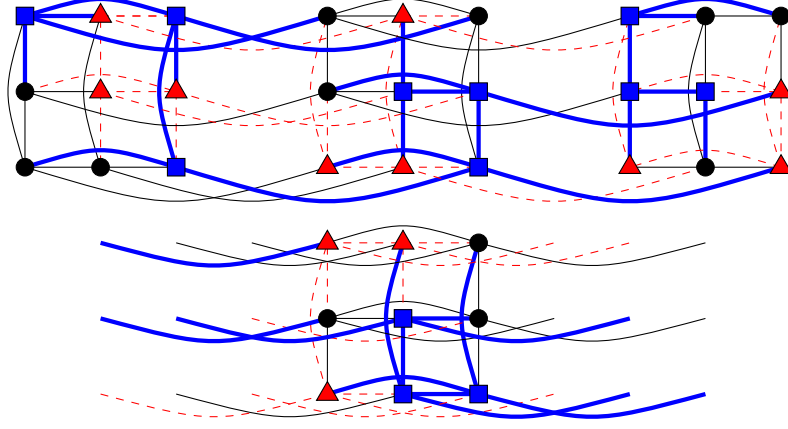


Figure 10: The three completely independent spanning trees on the last four levels of  $TM(3, 3, q)$ , for  $q \equiv 1 \pmod{3}$  and  $q > 2$ .

$TM(3p, 3p', 3q)$ , i.e. every edge is different for each value of  $i$ ,  $0 \leq i \leq pp'q - 1$ . Figure 9 describes the pattern on three levels for these three spanning trees for  $p = 1$  and  $p' = 1$ .

□

**Proposition 4.2.** *For any integer  $q \geq 3$ , there exists three completely independent spanning trees in  $TM(3, 3, q)$ .*

*Proof.* First, if  $q \equiv 0 \pmod{3}$ , then Proposition 4.1 allows us to conclude. For  $q \equiv 1 \pmod{3}$  ( $q \equiv 2 \pmod{3}$ , respectively), we define three completely independent spanning trees by using the pattern of Proposition 4.1 for every level except the last four (five, respectively) ones. If  $q \equiv 1 \pmod{3}$ , the trees are completed on the last four levels as depicted in Figure 10 (the corresponding edge sets are given in Appendix B.1). If  $q \equiv 2 \pmod{3}$ , the trees are completed on the last five levels as depicted in Figure 11 (the corresponding edge sets are given in Appendix B.2).

□

## 5 Conclusion

We conclude this paper by listing a few open problems:

1. Determine conditions which ensure that there exist  $r$  completely independent spanning trees in a graph.
2. Does any  $2r$ -connected graph with sufficiently large girth admit  $r$  completely independent spanning trees?



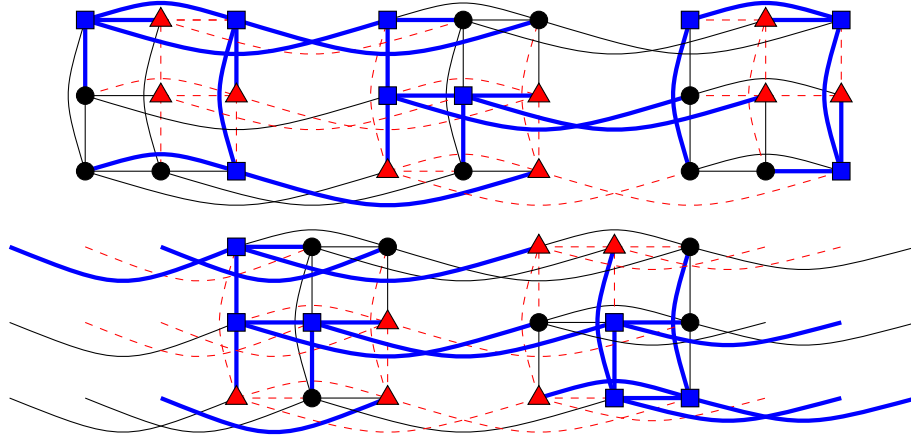


Figure 11: The three completely independent spanning trees on the last five levels of  $TM(3, 3, q)$ , for  $q \equiv 2 \pmod{3}$  and  $q > 2$ .

3. Is it true that in every 4-regular graph which is 4-connected, there exist 2 completely independent spanning trees?
4. Does the 6-dimensional hypercube  $Q_6 = C_4 \square C_4 \square C_4$  admit 3 completely independent spanning trees?

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## A Edge sets of the trees from Section 3

### A.1 Three completely independent spanning trees in $K_5 \square C_4$

$$\begin{aligned} E(T_1) &= \{u_0^0 u_3^0, u_0^0 u_2^0, u_3^0 u_1^0, u_3^0 u_4^0, u_1^0 u_1^1, u_3^0 u_4^1, u_2^0 u_1^2, u_2^0 u_4^2, u_0^3 u_2^3, u_0^3 u_3^3, u_2^3 u_1^3, u_2^3 u_4^3, \\ &u_0^0 u_1^0, u_3^0 u_3^1, u_1^0 u_2^2, u_3^0 u_3^2, u_0^2 u_0^3, u_2^2 u_2^3, u_0^3 u_0^4\}; \\ E(T_2) &= \{u_0^0 u_0^0, u_0^0 u_4^0, u_1^0 u_4^1, u_0^1 u_2^1, u_0^1 u_3^1, u_4^1 u_1^1, u_3^2 u_4^2, u_3^2 u_1^2, u_3^2 u_2^2, u_4^2 u_0^2, u_3^3 u_3^3, u_1^3 u_0^3, \\ &u_1^3 u_4^3, u_3^3 u_2^3, u_4^3 u_4^1, u_4^3 u_4^2, u_3^3 u_3^3, u_1^3 u_4^4, u_3^3 u_3^4\}; \\ E(T_3) &= \{u_0^0 u_0^0, u_0^0 u_3^0, u_0^0 u_4^0, u_0^0 u_1^0, u_1^0 u_2^1, u_1^0 u_0^1, u_2^1 u_3^1, u_2^1 u_4^1, u_0^2 u_1^2, u_0^2 u_2^2, u_0^2 u_3^2, u_1^2 u_4^2, \\ &u_4^3 u_0^3, u_4^3 u_3^3, u_2^3 u_1^3, u_1^3 u_1^2, u_1^3 u_2^2, u_2^3 u_2^3, u_3^3 u_4^3\}. \end{aligned}$$

### A.2 Three completely independent spanning trees in $K_5 \square C_5$

$$\begin{aligned} E(T_1) &= \{u_0^0 u_3^0, u_0^0 u_2^0, u_3^0 u_1^0, u_3^0 u_4^0, u_0^1 u_3^1, u_0^1 u_2^1, u_0^1 u_4^1, u_3^1 u_1^1, u_2^2 u_0^2, u_2^2 u_1^2, u_2^2 u_4^2, u_2^3 u_3^3, u_2^3 u_0^3, \\ &u_3^4 u_3^1, u_4^4 u_3^3, u_0^4 u_4^4, u_4^4 u_4^1, u_2^4 u_4^4, u_0^0 u_0^0, u_3^0 u_3^2, u_2^0 u_2^3, u_4^2 u_4^3, u_2^3 u_4^4, u_0^4 u_0^5\}; \\ E(T_2) &= \{u_0^0 u_0^0, u_0^0 u_2^0, u_4^1 u_1^1, u_4^1 u_3^1, u_0^2 u_2^2, u_0^2 u_3^2, u_4^2 u_1^2, u_4^2 u_2^2, u_0^3 u_3^3, u_0^3 u_4^3, u_3^3 u_1^3, u_3^3 u_2^3, \\ &u_4^4 u_4^4, u_4^4 u_4^2, u_4^4 u_4^1, u_4^4 u_0^4, u_0^4 u_1^4, u_0^4 u_4^4, u_0^4 u_0^2, u_4^4 u_4^2, u_0^4 u_3^3, u_3^4 u_4^4, u_4^4 u_1^5, u_4^4 u_5^5\}; \\ E(T_3) &= \{u_0^0 u_4^0, u_0^0 u_3^0, u_4^0 u_0^0, u_4^0 u_1^0, u_1^0 u_2^1, u_1^0 u_0^1, u_1^0 u_4^1, u_2^1 u_3^1, u_2^1 u_2^2, u_1^2 u_0^2, u_2^2 u_2^2, u_3^2 u_2^2, u_3^2 u_4^2, \\ &u_3^3 u_0^3, u_3^3 u_2^3, u_4^4 u_4^0, u_4^4 u_3^3, u_0^4 u_2^4, u_1^4 u_1^2, u_2^4 u_3^3, u_3^4 u_3^3, u_1^4 u_4^4, u_4^4 u_4^4, u_4^4 u_2^5, u_4^4 u_5^5\}. \end{aligned}$$

### A.3 Four completely independent spanning trees in $K_7 \square C_3$

$$\begin{aligned} E(T_1) &= \{u_0^0 u_1^0, u_0^0 u_3^0, u_0^0 u_5^0, u_0^0 u_6^0, u_0^1 u_2^1, u_1^0 u_4^1, u_0^1 u_5^1, u_2^1 u_1^1, u_2^1 u_3^1, u_2^1 u_6^1, u_2^2 u_4^2, u_2^2 u_5^2, \\ &u_2^2 u_6^2, u_2^3 u_0^3, u_2^3 u_1^3, u_4^2 u_2^3, u_0^0 u_0^1, u_2^0 u_2^1, u_0^2 u_2^2, u_4^0 u_4^2\}; \\ E(T_2) &= \{u_0^0 u_2^0, u_0^0 u_3^0, u_1^0 u_5^0, u_2^0 u_0^0, u_2^0 u_4^0, u_2^0 u_6^0, u_6^1 u_0^1, u_6^1 u_3^1, u_6^1 u_4^1, u_6^1 u_5^1, u_1^2 u_0^2, u_1^2 u_2^2, \\ &u_1^2 u_3^2, u_1^2 u_6^2, u_2^3 u_4^3, u_2^3 u_5^3, u_0^1 u_1^1, u_0^2 u_1^2, u_6^2 u_6^2, u_0^0 u_1^1\}; \\ E(T_3) &= \{u_0^3 u_2^0, u_0^3 u_0^0, u_3^0 u_5^0, u_4^0 u_0^0, u_4^0 u_1^0, u_4^0 u_6^0, u_1^1 u_0^1, u_1^1 u_3^1, u_1^1 u_4^1, u_1^1 u_6^1, u_4^1 u_2^1, u_4^1 u_5^1, \\ &u_3^2 u_0^2, u_3^2 u_2^2, u_3^2 u_5^2, u_3^2 u_6^2, u_4^4 u_4^1, u_1^4 u_1^2, u_4^4 u_4^2, u_3^0 u_3^3\}; \\ E(T_4) &= \{u_5^0 u_2^0, u_5^0 u_4^0, u_5^0 u_6^0, u_6^0 u_1^0, u_6^0 u_3^0, u_3^1 u_0^1, u_3^1 u_4^1, u_3^1 u_5^1, u_5^1 u_1^1, u_5^1 u_2^1, u_0^2 u_2^2, u_0^2 u_5^2, \\ &u_0^2 u_6^2, u_5^2 u_1^2, u_5^2 u_4^2, u_0^3 u_5^3, u_6^3 u_6^1, u_3^3 u_2^3, u_5^3 u_5^3, u_0^0 u_0^2\}. \end{aligned}$$

### A.4 Four completely independent spanning trees in $K_7 \square C_4$

$$\begin{aligned} E(T_1) &= \{u_0^0 u_1^0, u_0^0 u_3^0, u_0^0 u_5^0, u_0^0 u_6^0, u_0^1 u_1^1, u_0^1 u_3^1, u_0^1 u_2^1, u_0^1 u_4^1, u_2^1 u_1^1, u_2^1 u_5^1, u_2^2 u_3^2, u_2^2 u_5^2, u_2^2 u_6^2, \\ &u_5^2 u_0^2, u_5^2 u_1^2, u_5^2 u_4^2, u_4^3 u_3^3, u_4^3 u_5^3, u_4^3 u_6^3, u_5^3 u_0^3, u_5^3 u_1^3, u_0^0 u_0^1, u_2^0 u_2^1, u_2^0 u_2^2, u_2^0 u_3^2, u_5^0 u_5^3, u_4^0 u_4^3\}; \\ E(T_2) &= \{u_1^0 u_2^0, u_1^0 u_4^0, u_1^0 u_5^0, u_2^0 u_0^0, u_2^0 u_3^0, u_2^0 u_6^0, u_1^1 u_2^1, u_1^1 u_4^1, u_1^1 u_6^1, u_6^1 u_0^1, u_6^1 u_3^1, u_6^1 u_5^1, u_4^2 u_2^2, \\ &u_4^2 u_3^2, u_4^2 u_6^2, u_6^2 u_0^2, u_6^2 u_5^2, u_2^3 u_0^3, u_2^3 u_3^3, u_2^3 u_4^3, u_2^3 u_5^3, u_1^0 u_1^1, u_1^1 u_1^2, u_6^1 u_6^2, u_6^2 u_6^3, u_0^1 u_1^3, u_2^0 u_2^3\}; \\ E(T_3) &= \{u_3^0 u_4^0, u_3^0 u_1^0, u_3^0 u_5^0, u_4^0 u_0^0, u_4^0 u_2^0, u_4^0 u_6^0, u_4^1 u_2^1, u_4^1 u_5^1, u_4^1 u_6^1, u_5^1 u_0^1, u_5^1 u_1^1, u_1^2 u_0^2, u_1^2 u_2^2, \\ &u_1^2 u_4^2, u_1^2 u_6^2, u_1^3 u_0^3, u_1^3 u_2^3, u_1^3 u_3^3, u_1^3 u_4^3, u_3^3 u_5^3, u_3^3 u_7^3, u_3^0 u_3^1, u_4^0 u_4^1, u_5^0 u_5^2, u_1^1 u_1^3, u_2^3 u_3^3, u_3^0 u_3^3\}; \\ E(T_4) &= \{u_5^0 u_2^0, u_5^0 u_4^0, u_5^0 u_6^0, u_6^0 u_1^0, u_6^0 u_3^0, u_3^1 u_1^1, u_3^1 u_2^1, u_3^1 u_4^1, u_3^1 u_5^1, u_0^2 u_2^2, u_0^2 u_3^2, u_0^2 u_4^2, u_3^2 u_1^2, \\ &u_3^2 u_5^2, u_3^2 u_6^2, u_0^3 u_3^3, u_0^3 u_4^3, u_0^3 u_6^3, u_6^3 u_1^3, u_6^3 u_2^3, u_6^3 u_5^3, u_6^0 u_6^1, u_0^1 u_0^2, u_3^1 u_3^2, u_0^0 u_0^3, u_0^0 u_0^6\}. \end{aligned}$$

### A.5 Five completely independent spanning trees in $K_9 \square C_4$

$$\begin{aligned} E(T_1) &= \{u_0^0 u_2^0, u_0^0 u_4^0, u_0^0 u_5^0, u_0^0 u_8^0, u_5^0 u_1^0, u_5^0 u_3^0, u_5^0 u_6^0, u_5^0 u_7^0, u_0^1 u_2^1, u_0^1 u_4^1, u_0^1 u_6^1, u_0^1 u_7^1, u_4^1 u_1^1, \\ &u_4^1 u_3^1, u_1^1 u_5^1, u_4^1 u_8^1, u_0^2 u_2^2, u_0^2 u_4^2, u_0^2 u_6^2, u_4^2 u_1^2, u_4^2 u_8^2, u_8^2 u_2^2, u_8^2 u_5^2, u_8^2 u_7^2, u_8^3 u_3^1, u_8^3 u_2^2, u_8^3 u_3^3, u_8^3 u_6^3, \\ &u_8^3 u_7^3, u_0^0 u_0^1, u_0^0 u_0^2, u_4^4 u_4^3, u_8^2 u_8^3, u_0^0 u_0^3, u_5^0 u_5^3\}; \end{aligned}$$

$$\begin{aligned}
E(T_2) &= \{u_3^0u_0^0, u_3^0u_4^0, u_3^0u_7^0, u_3^0u_8^0, u_4^0u_1^0, u_4^0u_2^0, u_4^0u_5^0, u_4^0u_6^0, u_1^1u_0^1, u_1^1u_2^1, u_1^1u_6^1, u_1^1u_7^1, u_1^1u_8^1, \\
&u_1^2u_0^2, u_1^2u_2^2, u_1^2u_5^2, u_1^2u_7^2, u_1^2u_8^2, u_5^2u_4^2, u_5^2u_6^2, u_3^3u_2^3, u_3^3u_5^3, u_3^3u_6^3, u_3^3u_7^3, u_5^3u_0^3, u_5^3u_1^3, u_5^3u_4^3, u_5^3u_8^3, \\
&u_3^0u_3^1, u_4^0u_4^1, u_1^1u_1^2, u_5^1u_5^2, u_3^2u_3^3, u_5^2u_5^3, u_3^0u_3^3\}; \\
E(T_3) &= \{u_2^0u_1^0, u_2^0u_3^0, u_2^0u_5^0, u_2^0u_7^0, u_7^0u_0^0, u_7^0u_4^0, u_7^0u_6^0, u_7^0u_8^0, u_2^1u_3^1, u_2^1u_4^1, u_2^1u_5^1, u_2^1u_6^1, u_3^1u_0^1, \\
&u_3^1u_1^1, u_3^1u_7^1, u_3^1u_8^1, u_3^2u_1^2, u_3^2u_4^2, u_3^2u_5^2, u_3^2u_6^2, u_6^2u_8^2, u_3^3u_2^3, u_3^3u_3^3, u_3^3u_4^3, u_3^3u_7^3, u_3^3u_8^3, u_7^3u_1^3, u_7^3u_5^3, \\
&u_7^3u_6^3, u_2^0u_2^1, u_2^1u_2^2, u_3^1u_3^2, u_2^0u_3^3, u_7^2u_7^3, u_7^0u_7^3\}; \\
E(T_4) &= \{u_6^0u_0^0, u_6^0u_1^0, u_6^0u_2^0, u_6^0u_3^0, u_6^0u_8^0, u_5^1u_0^1, u_5^1u_1^1, u_5^1u_5^1, u_5^1u_7^1, u_5^1u_8^1, u_7^1u_2^1, u_7^1u_4^1, u_7^1u_6^1, \\
&u_6^2u_1^2, u_6^2u_2^2, u_6^2u_4^2, u_6^2u_7^2, u_6^2u_8^2, u_7^2u_0^2, u_7^2u_3^2, u_7^2u_5^2, u_4^3u_2^3, u_4^3u_3^3, u_4^3u_6^3, u_4^3u_7^3, u_4^3u_8^3, u_6^3u_0^3, u_6^3u_1^3, \\
&u_6^3u_5^3, u_5^0u_5^1, u_7^0u_7^1, u_7^1u_7^2, u_6^2u_6^3, u_4^0u_4^3, u_6^0u_6^3\}; \\
E(T_5) &= \{u_1^0u_0^0, u_1^0u_3^0, u_1^0u_7^0, u_1^0u_8^0, u_8^0u_2^0, u_8^0u_4^0, u_8^0u_5^0, u_6^1u_1^1, u_6^1u_4^1, u_6^1u_5^1, u_6^1u_8^1, u_8^1u_0^1, u_8^1u_2^1, \\
&u_8^1u_7^1, u_2^2u_0^2, u_2^2u_3^2, u_2^2u_4^2, u_2^2u_5^2, u_2^2u_7^2, u_1^3u_0^3, u_1^3u_2^3, u_1^3u_3^3, u_1^3u_4^3, u_2^3u_3^3, u_2^3u_6^3, u_2^3u_7^3, u_1^0u_1^1, u_6^0u_6^1, \\
&u_8^0u_8^1, u_6^1u_6^2, u_8^1u_8^2, u_1^1u_1^2, u_2^1u_2^3, u_1^0u_1^3, u_8^0u_8^3\}.
\end{aligned}$$

## A.6 Five completely independent spanning trees in $K_9 \square C_4$

$$\begin{aligned}
E(T_1) &= \{u_0^0u_2^0, u_0^0u_3^0, u_0^0u_5^0, u_0^0u_6^0, u_0^0u_7^0, u_0^0u_1^0, u_6^0u_4^0, u_6^0u_8^0, u_6^1u_1^1, u_6^1u_2^1, u_6^1u_5^1, u_6^1u_7^1, u_6^1u_8^1, \\
&u_3^2u_0^2, u_3^2u_1^2, u_3^2u_4^2, u_3^2u_6^2, u_3^2u_8^2, u_4^2u_2^2, u_4^2u_5^2, u_4^2u_7^2, u_4^3u_0^3, u_4^3u_3^3, u_4^3u_6^3, u_4^3u_7^3, u_4^3u_8^3, u_7^2u_3^2, u_7^2u_5^2, \\
&u_4^4u_0^4, u_4^4u_2^4, u_4^4u_6^4, u_4^4u_8^4, u_6^4u_4^4, u_6^4u_3^4, u_6^4u_5^4, u_0^0u_0^1, u_0^0u_6^1, u_3^1u_3^2, u_4^1u_4^2, u_4^2u_4^3, u_4^3u_4^4, u_6^2u_6^3, u_6^3u_6^4, u_6^0u_6^4\}; \\
E(T_2) &= \{u_7^0u_1^0, u_7^0u_2^0, u_7^0u_4^0, u_7^0u_6^0, u_7^0u_8^0, u_3^1u_1^1, u_3^1u_6^1, u_3^1u_7^1, u_3^1u_8^1, u_7^1u_0^1, u_7^1u_2^1, u_7^1u_6^1, \\
&u_0^2u_1^2, u_0^2u_4^2, u_0^2u_5^2, u_0^2u_7^2, u_0^2u_8^2, u_7^2u_2^2, u_7^2u_3^2, u_7^2u_6^2, u_3^3u_1^3, u_3^3u_6^3, u_3^3u_7^3, u_3^3u_8^3, u_7^3u_0^3, u_7^3u_2^3, u_7^3u_4^3, u_7^3u_8^3, \\
&u_0^4u_1^4, u_0^4u_5^4, u_0^4u_6^4, u_0^4u_7^4, u_5^4u_2^4, u_5^4u_3^4, u_5^4u_4^4, u_5^4u_8^4, u_3^0u_3^1, u_0^1u_7^1, u_7^1u_7^2, u_0^2u_0^3, u_7^2u_7^3, u_3^0u_0^4, u_0^0u_0^4, u_0^0u_5^4\}; \\
E(T_3) &= \{u_4^0u_0^0, u_4^0u_3^0, u_4^0u_7^0, u_4^0u_8^0, u_7^0u_1^0, u_7^0u_2^0, u_7^0u_5^0, u_7^0u_6^0, u_2^1u_1^1, u_2^1u_3^1, u_2^1u_4^1, u_2^1u_5^1, u_2^1u_8^1, \\
&u_1^1u_0^1, u_1^1u_6^1, u_1^1u_7^1, u_1^2u_2^2, u_1^2u_4^2, u_1^2u_5^2, u_1^2u_7^2, u_1^2u_8^2, u_2^2u_0^2, u_2^2u_3^2, u_2^2u_6^2, u_2^3u_0^3, u_2^3u_3^3, u_2^3u_6^3, u_2^3u_7^3, u_2^3u_8^3, \\
&u_5^3u_1^3, u_5^3u_4^3, u_5^3u_6^3, u_5^3u_8^3, u_4^4u_0^4, u_4^4u_1^4, u_4^4u_3^4, u_4^4u_4^4, u_4^4u_6^4, u_4^4u_8^4, u_0^1u_0^2, u_0^1u_1^2, u_2^3u_3^3, u_2^3u_8^3, u_3^3u_4^3, u_0^0u_4^4, u_0^0u_7^4\}; \\
E(T_4) &= \{u_1^0u_0^0, u_1^0u_3^0, u_1^0u_4^0, u_1^0u_8^0, u_3^0u_2^0, u_3^0u_5^0, u_3^0u_6^0, u_3^0u_7^0, u_0^1u_1^1, u_0^1u_2^1, u_0^1u_3^1, u_0^1u_5^1, u_0^1u_6^1, \\
&u_1^1u_4^1, u_1^1u_7^1, u_1^1u_8^1, u_8^2u_2^2, u_8^2u_4^2, u_8^2u_5^2, u_8^2u_6^2, u_8^2u_7^2, u_3^3u_0^3, u_3^3u_3^3, u_3^3u_6^3, u_3^3u_7^3, u_3^3u_8^3, u_8^3u_1^3, u_8^3u_2^3, u_8^3u_4^3, \\
&u_8^3u_6^3, u_4^3u_0^3, u_4^3u_4^3, u_4^3u_7^3, u_4^3u_8^3, u_7^4u_2^4, u_7^4u_5^4, u_7^4u_6^4, u_0^0u_0^1, u_0^0u_2^1, u_1^1u_1^2, u_2^3u_3^3, u_8^3u_8^3, u_3^3u_4^3, u_0^0u_4^4, u_0^0u_7^4\}; \\
E(T_5) &= \{u_2^0u_1^0, u_2^0u_4^0, u_2^0u_6^0, u_2^0u_8^0, u_8^0u_0^0, u_8^0u_3^0, u_8^0u_7^0, u_5^1u_1^1, u_5^1u_3^1, u_5^1u_4^1, u_5^1u_8^1, u_8^1u_0^1, u_8^1u_7^1, \\
&u_5^2u_2^2, u_5^2u_3^2, u_5^2u_6^2, u_5^2u_7^2, u_6^2u_0^2, u_6^2u_1^2, u_6^2u_4^2, u_6^2u_8^2, u_3^3u_0^3, u_3^3u_3^3, u_3^3u_6^3, u_3^3u_7^3, u_4^1u_4^3, u_4^1u_4^4, u_4^1u_5^4, \\
&u_4^1u_8^4, u_8^4u_0^4, u_8^4u_6^4, u_8^4u_7^4, u_2^0u_2^1, u_5^0u_5^1, u_8^0u_8^1, u_5^1u_5^2, u_6^1u_6^2, u_8^1u_8^2, u_5^2u_5^3, u_6^2u_6^3, u_7^3u_4^3, u_8^3u_8^3, u_2^0u_2^4, u_8^0u_8^4\}.
\end{aligned}$$

## B Edge sets of the trees from Section 4

### B.1 Three completely independent spanning trees in the last four levels of $TM(3, 3, q)$

$$\begin{aligned}
E(T_1) &= \{(0, 0, 0)(1, 0, 0), (0, 0, 0)(0, 1, 0), (0, 0, 0)(0, 2, 0), (0, 0, 0)(0, 0, 1), \\
&(1, 0, 0)(1, 2, 0), (1, 0, 0)(2, 0, 0), (1, 0, 0)(1, 0, 1), (0, 1, 0)(1, 1, 0), (0, 1, 0)(0, 1, 1), \\
&(0, 1, 1)(1, 1, 1), (0, 1, 1)(0, 2, 1), (0, 1, 1)(0, 1, 2), (0, 2, 1)(1, 2, 1), (0, 2, 1)(2, 2, 1), \\
&(0, 2, 1)(0, 2, 2), (2, 2, 1)(2, 0, 1), (2, 2, 1)(2, 1, 1), (1, 0, 2)(0, 0, 2), (1, 0, 2)(2, 0, 2), \\
&(1, 0, 2)(1, 2, 2), (1, 0, 2)(1, 0, 3), (1, 2, 2)(2, 2, 2), (1, 2, 2)(1, 1, 2), (1, 2, 2)(1, 2, 3), \\
&(2, 2, 2)(2, 1, 2), (2, 2, 2)(2, 2, 3), (0, 1, 3)(1, 1, 3), (0, 1, 3)(0, 0, 3), (0, 1, 3)(2, 1, 3), \\
&(0, 1, 3)(0, 1, 4), (2, 1, 3)(2, 0, 3), (2, 1, 3)(2, 2, 3), (2, 1, 3)(2, 1, 4), (2, 2, 3)(0, 2, 3), \\
&(2, 2, 3)(2, 2, 4)\}; \\
E(T_2) &= \{(1, 1, 0)(2, 1, 0), (1, 1, 0)(1, 0, 0), (1, 1, 0)(1, 2, 0), (1, 1, 0)(1, 1, 1),
\end{aligned}$$

$(2, 1, 0)(0, 1, 0), (2, 1, 0)(2, 0, 0), (2, 1, 0)(2, 1, 1), (1, 2, 0)(2, 2, 0), (1, 2, 0)(1, 2, 1),$   
 $(0, 0, 1)(1, 0, 1), (0, 0, 1)(0, 1, 1), (0, 0, 1)(0, 2, 1), (1, 0, 1)(2, 0, 1), (1, 0, 1)(1, 2, 1),$   
 $(1, 0, 1)(1, 0, 2), (1, 2, 1)(2, 2, 1), (1, 2, 1)(1, 2, 2), (0, 0, 2)(2, 0, 2), (0, 0, 2)(0, 2, 2),$   
 $(0, 0, 2)(0, 0, 3), (2, 0, 2)(2, 1, 2), (2, 0, 2)(2, 2, 2), (2, 0, 2)(2, 0, 3), (2, 1, 2)(0, 1, 2),$   
 $(2, 1, 2)(1, 1, 2), (2, 1, 2)(2, 1, 3), (0, 0, 3)(1, 0, 3), (0, 0, 3)(0, 2, 3), (0, 0, 3)(0, 0, 4),$   
 $(0, 2, 3)(1, 2, 3), (0, 2, 3)(0, 1, 3), (0, 2, 3)(0, 2, 4), (1, 2, 3)(2, 2, 3), (1, 2, 3)(1, 1, 3),$   
 $(1, 2, 3)(1, 2, 4)\};$

$E(T_3) = \{(2, 0, 0)(0, 0, 0), (2, 0, 0)(2, 2, 0), (2, 0, 0)(2, 0, 1), (0, 2, 0)(1, 2, 0),$   
 $(0, 2, 0)(2, 2, 0), (0, 2, 0)(0, 1, 0), (0, 2, 0)(0, 2, 1), (2, 2, 0)(2, 1, 0), (2, 2, 0)(2, 2, 1),$   
 $(2, 0, 1)(0, 0, 1), (2, 0, 1)(2, 1, 1), (2, 0, 1)(2, 0, 2), (1, 1, 1)(2, 1, 1), (1, 1, 1)(1, 0, 1),$   
 $(1, 1, 1)(1, 2, 1), (2, 1, 1)(0, 1, 1), (2, 1, 1)(2, 1, 2), (0, 1, 2)(1, 1, 2), (0, 1, 2)(0, 0, 2),$   
 $(0, 1, 2)(0, 2, 2), (0, 1, 2)(0, 1, 3), (1, 1, 2)(1, 0, 2), (1, 1, 2)(1, 1, 3), (0, 2, 2)(1, 2, 2),$   
 $(0, 2, 2)(2, 2, 2), (0, 2, 2)(0, 2, 3), (1, 0, 3)(2, 0, 3), (1, 0, 3)(1, 1, 3), (1, 0, 3)(1, 2, 3),$   
 $(1, 0, 3)(1, 0, 4), (2, 0, 3)(0, 0, 3), (2, 0, 3)(2, 2, 3), (2, 0, 3)(2, 0, 4), (1, 1, 3)(2, 1, 3),$   
 $(1, 1, 3)(1, 1, 4)\}.$

## B.2 Three completely independent spanning trees in the last five levels of $TM(3, 3, q)$

$E(T_1) = \{(0, 0, 0)(1, 0, 0), (0, 0, 0)(0, 1, 0), (0, 0, 0)(0, 2, 0), (0, 0, 0)(0, 0, 1),$   
 $(1, 0, 0)(1, 2, 0), (1, 0, 0)(2, 0, 0), (1, 0, 0)(1, 0, 1), (0, 1, 0)(1, 1, 0), (0, 1, 0)(0, 1, 1),$   
 $(1, 0, 1)(2, 0, 1), (1, 0, 1)(1, 2, 1), (1, 2, 1)(2, 2, 1), (1, 2, 1)(1, 1, 1), (1, 2, 1)(1, 2, 2),$   
 $(2, 2, 1)(0, 2, 1), (2, 2, 1)(2, 1, 1), (2, 2, 1)(2, 2, 2), (0, 0, 2)(1, 0, 2), (0, 0, 2)(2, 0, 2),$   
 $(0, 0, 2)(0, 1, 2), (0, 0, 2)(0, 0, 3), (1, 0, 2)(1, 1, 2), (1, 0, 2)(1, 0, 3), (0, 1, 2)(2, 1, 2),$   
 $(0, 1, 2)(0, 2, 2), (0, 1, 2)(0, 1, 3), (1, 0, 3)(2, 0, 3), (1, 0, 3)(1, 2, 3), (1, 0, 3)(1, 0, 4),$   
 $(1, 2, 3)(2, 2, 3), (1, 2, 3)(1, 1, 3), (1, 2, 3)(1, 2, 4), (2, 2, 3)(0, 2, 3), (2, 2, 3)(2, 1, 3),$   
 $(2, 2, 3)(2, 2, 4), (0, 1, 4)(1, 1, 4), (0, 1, 4)(0, 0, 4), (0, 1, 4)(2, 1, 4), (0, 1, 4)(0, 1, 5),$   
 $(2, 1, 4)(2, 0, 4), (2, 1, 4)(2, 2, 4), (2, 1, 4)(2, 1, 5), (2, 2, 4)(0, 2, 4), (2, 2, 4)(2, 2, 5)\};$

$E(T_2) = \{(1, 1, 0)(2, 1, 0), (1, 1, 0)(1, 0, 0), (1, 1, 0)(1, 2, 0), (1, 1, 0)(1, 1, 1),$   
 $(2, 1, 0)(0, 1, 0), (2, 1, 0)(2, 0, 0), (2, 1, 0)(2, 1, 1), (1, 2, 0)(2, 2, 0), (1, 2, 0)(1, 2, 1),$   
 $(0, 0, 1)(1, 0, 1), (0, 0, 1)(2, 0, 1), (0, 0, 1)(0, 2, 1), (0, 0, 1)(0, 0, 2), (2, 0, 1)(2, 1, 1),$   
 $(2, 0, 1)(2, 2, 1), (2, 0, 1)(2, 0, 2), (2, 1, 1)(0, 1, 1), (1, 1, 2)(0, 1, 2), (1, 1, 2)(2, 1, 2),$   
 $(1, 1, 2)(1, 2, 2), (1, 1, 2)(1, 1, 3), (2, 1, 2)(2, 2, 2), (2, 1, 2)(2, 1, 3), (1, 2, 2)(0, 2, 2),$   
 $(1, 2, 2)(1, 0, 2), (1, 2, 2)(1, 2, 3), (0, 0, 3)(1, 0, 3), (0, 0, 3)(2, 0, 3), (0, 0, 3)(0, 2, 3),$   
 $(0, 0, 3)(0, 0, 4), (2, 0, 3)(2, 1, 3), (2, 0, 3)(2, 2, 3), (2, 0, 3)(2, 0, 4), (2, 1, 3)(0, 1, 3),$   
 $(2, 1, 3)(2, 1, 4), (0, 0, 4)(1, 0, 4), (0, 0, 4)(0, 2, 4), (0, 0, 4)(0, 0, 5), (0, 2, 4)(1, 2, 4),$   
 $(0, 2, 4)(0, 1, 4), (0, 2, 4)(0, 2, 5), (1, 2, 4)(2, 2, 4), (1, 2, 4)(1, 1, 4), (1, 2, 4)(1, 2, 5)\};$

$E(T_3) = \{(2, 0, 0)(0, 0, 0), (2, 0, 0)(2, 2, 0), (2, 0, 0)(2, 0, 1), (0, 2, 0)(1, 2, 0),$   
 $(0, 2, 0)(2, 2, 0), (0, 2, 0)(0, 1, 0), (0, 2, 0)(0, 2, 1), (2, 2, 0)(2, 1, 0), (2, 2, 0)(2, 2, 1),$   
 $(0, 1, 1)(1, 1, 1), (0, 1, 1)(0, 0, 1), (0, 1, 1)(0, 2, 1), (0, 1, 1)(0, 1, 2), (1, 1, 1)(2, 1, 1),$   
 $(1, 1, 1)(1, 0, 1), (1, 1, 1)(1, 1, 2), (0, 2, 1)(1, 2, 1), (2, 0, 2)(1, 0, 2), (2, 0, 2)(2, 1, 2),$   
 $(2, 0, 2)(2, 2, 2), (2, 0, 2)(2, 0, 3), (0, 2, 2)(2, 2, 2), (0, 2, 2)(0, 0, 2), (0, 2, 2)(0, 2, 3),$   
 $(2, 2, 2)(1, 2, 2), (2, 2, 2)(2, 2, 3), (0, 1, 3)(1, 1, 3), (0, 1, 3)(0, 0, 3), (0, 1, 3)(0, 2, 3),$   
 $(0, 1, 3)(0, 1, 4), (1, 1, 3)(2, 1, 3), (1, 1, 3)(1, 0, 3), (1, 1, 3)(1, 1, 4), (0, 2, 3)(1, 2, 3),$   
 $(0, 2, 3)(0, 2, 4), (1, 0, 4)(2, 0, 4), (1, 0, 4)(1, 1, 4), (1, 0, 4)(1, 2, 4), (1, 0, 4)(1, 0, 5),$   
 $(2, 0, 4)(0, 0, 4), (2, 0, 4)(2, 2, 4), (2, 0, 4)(2, 0, 5), (1, 1, 4)(2, 1, 4), (1, 1, 4)(1, 1, 5)\}.$