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A characterization of b-chromatic and partial Grundy numbers by induced subgraphs

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Abstract

Gyárfás et al. and Zaker have proven that the Grundy number of a graph $G$ satisfies $\Gamma(G) \geq t$ if and only if $G$ contains an induced subgraph called a $t$-atom. The family of $t$-atoms has bounded order and contains a finite number of graphs. In this article, we introduce equivalents of $t$-atoms for b-coloring and partial Grundy coloring. This concept is used to prove that determining if $\varphi(G) \geq t$ and $\partial \Gamma(G) \geq t$ (under conditions for the b-coloring), for a graph $G$, is in XP with parameter $t$. We illustrate the utility of the concept of $t$-atoms by giving results on b-critical vertices and edges, on b-perfect graphs and on graphs of girth at least 7.

1 Introduction

Given a graph $G$, a proper $k$-coloring of $G$ is a surjective function $c : V(G) \rightarrow \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for any $uv \in E(G)$; the color class $V_i$ is the set $\{u \in V | c(u) = i\}$ and a vertex $v$ has color $i$ if $v \in V_i$. We denote by $N(u)$ the set of neighbors of a vertex $u$ and by $N[u]$ the set $N(u) \cup \{u\}$. A vertex $v$ of color $i$ is a Grundy vertex if it is adjacent to at least one vertex colored $j$, for every $j < i$. A Grundy $k$-coloring is a proper $k$-coloring such that every vertex is a Grundy vertex. The Grundy number of a graph $G$, denoted by $\Gamma(G)$, is the largest integer $k$ such that there exists a Grundy $k$-coloring of $G$ [10]. A partial Grundy $k$-coloring is a proper $k$-coloring such that every color class contains at least one Grundy vertex. The partial Grundy number of a graph $G$, denoted by $\partial \Gamma(G)$, is the largest integer $k$ such that there exists a partial Grundy $k$-coloring of $G$. Let $G$ and $G'$ be two graphs. By $G \cup G'$ we denote the graph with vertex set $V(G) \cup V(G')$ and edge set $E(G) \cup E(G')$. Let $m(G)$ be the largest integer
Figure 1: The graph $K_{3,3}$ with $\varphi(K_{3,3}) = 2$ (on the left) and $\varphi_r(K_{3,3}) = 3$ (on the right).

$m$ such that $G$ has at least $m$ vertices of degree at least $m - 1$. A graph $G$ is tight if it has exactly $m(G)$ vertices of degree $m(G) - 1$.

Another coloring parameter with domination constraints on the colors is the $b$-chromatic number. In a proper-$k$-coloring, a vertex $v$ of color $i$ is a $b$-vertex if $v$ is adjacent to at least one vertex colored $j$, $1 \leq j \neq i \leq k$. A $b$-$k$-coloring, also called $b$-coloring when $k$ is not specified, is a proper $k$-coloring such that every color class contains at least one $b$-vertex. The $b$-chromatic number of a graph $G$, denoted by $\varphi(G)$, is the largest integer $k$ such that there exists a $b$-$k$-coloring of $G$. In this paper, we introduce the concept of $b$-relaxed number, denoted by $\varphi_r(G)$. A $b$-$k$-relaxed coloring of $G$ is a $b$-$k$-coloring of a subgraph of $G$. The $b$-relaxed number of $G$ is $\varphi_r(G) = \max_{H \subseteq G} (\varphi(H))$, for $H$ an induced subgraph of $G$. Note that we have $\varphi(G) \leq \varphi_r(G) \leq \partial \Gamma(G)$. The difference between $\varphi(G)$ and $\varphi_r(G)$ can be arbitrary large. Let $K_{n,n}^-$ denotes the complete bipartite graph $K_{n,n}$ in which we remove $n - 1$ pairwise non incident edges (or $n - 1$ edges of a perfect matching in $K_{n,n}$) [1]. For this graph we have $\varphi(K_{n,n}^-) = 2$ and $\varphi_r(K_{n,n}^-) = n$ as Figure 1 illustrates it (for $n = 3$).

The concept of $b$-coloring has been introduced by Irving and Manlove [16], and a large number of papers was published (see e.g. [8, 19]). The $b$-chromatic number of regular graphs has been investigated in a serie of papers ([6, 17, 20, 22]). Determining the $b$-chromatic number of a tight graph is NP-hard even for a connected bipartite graph [18] and a tight chordal graph [12].

In this paper, we study the decision problems $b$-COL, $b$-r-COL and $pG$-COL with parameter $t$ from Table 1.

<table>
<thead>
<tr>
<th>Question</th>
<th>b-COL</th>
<th>b-r-COL</th>
<th>G-COL</th>
<th>pG-COL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complexity class</td>
<td>undetermined</td>
<td>XP</td>
<td>XP [23]</td>
<td>XP</td>
</tr>
</tbody>
</table>

Table 1: The different decision problems with input a graph $G$ and parameter $t$ and their complexity class.

A decision problem is in FPT with parameter $t$ if there exists an algorithm which resolves the problem in time $O(f(t) \ n^c)$, for an instance of size $n$, a computable
function $f$ and a constant $c$. A decision problem is in XP with parameter $t$ if there exists an algorithm which resolves the problem in time $O(f(t) n^{g(t)})$, for an instance of size $n$ and two computable functions $f$ and $g$.

The concept of $t$-atom was introduced independently by Gyárfás et al. [11] and by Zaker [23]. The family of $t$-atoms is finite and the presence of a $t$-atom can be determined in polynomial time for a fixed $t$. The following definition is slightly different from the definitions of Gyárfás et al. or Zaker, insisting more on the construction of every $t$-atom (some $t$-atoms cannot be obtained with the initial construction of Zaker).

**Definition 1.1** ([23]). The family of $t$-atoms is denoted by $A_t^Γ$, for $t ≥ 1$, and is defined by induction. The family $A_1^Γ$ only contains $K_1$. A graph $G$ is in $A_{t+1}^Γ$ if there exists a graph $G'$ in $A_t^Γ$ and an integer $m$, $m ≤ |V(G')|$, such that $G$ is composed of $G'$ and an independent set $I_m$ of order $m$, adding edges between $G'$ and $I_m$ such that every vertex in $G'$ is adjacent to at least one vertex in $I_m$.

Moreover, in the following sections, we say that a graph $G$ in a family of graphs $F$ is minimal, if no graphs of $F$ is a proper induced subgraph of $G$. For example, a minimal $t$-atom $A$ is a $t$-atom for which there are no $t$-atoms which are induced in $A$ other than itself.

**Theorem 1.1** ([11, 23]). A graph $G$ satisfies $Γ(G) ≥ t$ if and only if it contains an induced minimal $t$-atom.

In this paper we prove equivalent theorems for b-relaxed number and partial Grundy number. In contrast with the minimal $t$-atoms, we can not define the minimal $t$-atoms for b-coloring as the smallest graphs such that $G$ satisfies $ϕ(G) = t$ (also called b-critical graphs).

The paper is organized as follows: Section 2 is devoted to the definition of $t$-atoms for the partial Grundy coloring. This concept allows us to prove that the partial Grundy coloring problem is in XP with parameter $t$. Section 3 is similar to Section 2 but for $b$-relaxed-coloring. Section 4 is devoted to the concept of b-critical vertices and edges. Section 5 is about b-perfect graphs. Finally, Section 6 deals with graphs for which the b-relaxed and the b-chromatic numbers are equal.

## 2 Partial-Grundy-$t$-atoms: $t$-atoms for partial Grundy coloring

We start this section with the definition of $t$-atoms for partial Grundy coloring.

**Definition 2.1.** Given an integer $t$, a partial Grundy $t$-atom (or pG-$t$-atom, for short) is a graph $G$ whose vertex-set can be partitioned into $t$ sets $D_1, \ldots, D_t$, where $D_i$ contains a special vertex $c_i$ for each $i ∈ \{1, \ldots, t\}$ such that the following holds:

- For all $i ∈ \{1, \ldots, t\}$, $D_i$ is an independent set and $|D_i| ≤ t - i + 1$;
Figure 2: The minimal pG-2-atom (on the left) and the three minimal pG-3-atoms (the numbers are the colors of the vertices and the surrounded vertices form the centers).

- For all \( i \in \{2, \ldots, t\} \), \( c_i \) has a neighbor in each of \( D_1, \ldots, D_i-1 \).

The set \( \{c_1, \ldots, c_t\} \) is called the center of \( A \) and denoted by \( C(A) \).

Note that the sets \( D_1, \ldots, D_t \) induce a partial Grundy coloring of the pG-\( t \)-atom. Figure 2 illustrates several pG-\( t \)-atoms (and their induced colorings) obtained using the previous definition.

**Observation 2.1.** For every pG-\( t \)-atom \( G \), we have \( |V(G)| \leq \frac{t(t+1)}{2} \).

**Lemma 2.2.** Let \( t \) and \( t' \) be two integers such that \( 1 \leq t' < t \). Every pG-\( t \)-atom contains a pG-\( t' \)-atom as induced subgraph.

**Proof.** Every pG-\( t \)-atom \( G \) contains a pG-\( t' \)-atom \( G' \): we can obtain \( G' \) by removing every vertex in \( D_k \), for \( t' < k \leq t \), and by removing, afterwards, the vertices of \( G' \) not adjacent to any vertex in \( \{c_1, \ldots, c_{t'}\} \).

Note that the only minimal pG-2-atom is \( P_2 \). The minimal pG-3-atoms are \( C_3 \), \( P_4 \) and \( P_2 \cup P_3 \). These graphs are illustrated in Figure 2.

**Theorem 2.3.** For a graph \( G \), we have \( \partial \Gamma(G) \geq t \) if and only if \( G \) contains an induced minimal pG-\( t \)-atom.

**Proof.** Suppose that \( \partial \Gamma(G) = t' \) with \( t' \geq t \). By definition, there exists a partial Grundy coloring of \( G \) with \( t' \) colors. Let \( u_1, \ldots, u_{t'} \) be a set of Grundy vertices, each in a different color class of \( V(G) \). The graph induced by \( N[u_1] \cup \ldots \cup N[u_{t'}] \) contains a pG-\( t' \)-atom. Hence, by Lemma 2.2, since \( G \) contains an induced pG-\( t' \)-atom, then it also contains an induced minimal pG-\( t \)-atom.

Suppose \( G \) contains an induced minimal pG-\( t \)-atom. Thus, the sets \( D_1, \ldots, D_t \) induce a partial-Grundy coloring of this pG-\( t \)-atom. We can extend this coloring to a partial Grundy coloring of \( G \) with at least \( t \) colors in a greedy way by coloring the remaining vertices in any order, assigning to each of them the smallest color not used by its neighbors.
Proposition 2.4. Let $G$ be a graph of order $n$ and let $t$ be an integer. There exists an algorithm in time $O(n^{\frac{t(t+1)}{2}})$ to determine if $\partial \Gamma(G) \geq t$. Hence, the problem $pG$-COL with parameter $t$ is in XP.

Proof. By Theorem 2.3, it suffices to verify that $G$ contains an induced minimal $pG$-$t$-atom to have $\partial \Gamma(G) \geq t$. Since the order of a minimal $pG$-$t$-atom is bounded by $\frac{(t+1)^2}{2}$, we obtain an algorithm in time $O(n^{\frac{t(t+1)}{2}})$.

We finish this section by determining every graph $G$ with $\partial \Gamma(G) = 2$.

Proposition 2.5. For a graph $G$ without isolated vertices, we have $\partial \Gamma(G) = 2$ if and only if $G = K_{n,m}$, for $n \geq 2$ and $m \geq 1$ or $G$ only contains isolated edges.

Proof. Zaker [23] has proven that $\Gamma(G) = 2$ if and only if $G$ is the disjoint union of copies of some $K_{n,m}$, for $n \geq 1$ and $m \geq 1$. Let $n$ and $m$ be positive integers.

We can note that a graph containing a copy of $K_{n,m}$, for $n \geq 2$ and $m \geq 1$ and a copy of $K_{n,m}$, for $n \geq 1$ and $m \geq 1$ contains an induced $P_3 \cup P_2$, hence a pG-3-atom. Hence, if $\partial \Gamma(G) = 2$, then $G = K_{n,m}$, for $n \geq 2$ and $m \geq 1$ or $G$ only contains isolated edges.

Moreover, neither $K_{n,m}$ nor $P_2 \cup \ldots \cup P_2$ does contain an induced $C_3$, $P_4$ or $P_3 \cup P_2$. Hence, $\partial \Gamma(K_{n,m}) = 2$.

3 b-t-atoms: t-atoms for b-coloring

As in the previous section, we start this section with the definition of b-t-atoms (the notion of t-atom for b-coloring).

Definition 3.1. Given an integer $t$, a b-t-atom is a graph $A$ whose vertex-set can be partitioned into $t$ sets $D_1, \ldots, D_t$, where $D_i$ contains a special vertex $c_i$ for each $i \in \{1, \ldots, t\}$ such that the following holds:

- For each $i \in \{1, \ldots, t\}$, $D_i$ is an independent set and $|D_i| \leq t$;
- For all $i, j \in \{1, \ldots, t\}$, with $i \neq j$, $c_i$ has a neighbor in $D_j$.

The set $\{c_1, \ldots, c_t\}$ is called the center of $A$ and denoted by $C(A)$.

Note that the sets $D_1, \ldots, D_t$ induce a b-coloring of the b-t-atom. Figure 3 illustrates several b-t-atoms (and their induced coloring) obtained using the previous definition.

Observation 3.1. For every b-t-atom $G$, we have $|V(G)| \leq t^2$.

Lemma 3.2. Let $t$ and $t'$ be two integers such that $1 \leq t' < t$. Every b-t-atom contains a b-$t'$-atom as induced subgraph.

Proof. Every b-t-atom $G$ contains a b-$t'$-atom $G'$: we can obtain $G'$ by removing every vertex in $D_k$, for $t' < k \leq t$, and by removing, afterwards, the vertices not adjacent to any vertex in $\{c_1, \ldots, c_{t'}\}$.
Figure 3: The minimal b-2-atom (on the left) and the five minimal b-3-atoms.

Note that the only minimal b-2-atom is $P_2$. The minimal b-3-atoms are $C_3$, $P_5$, $C_5$, $P_3 \cup P_4$ and $P_3 \cup P_3 \cup P_3$. These graphs are illustrated in Figure 3.

**Observation 3.3.** Every minimal pG-$t$-atom is an induced subgraph of a minimal b-$t$-atom or a minimal $t$-atom (an atom for the Grundy number).

**Proposition 3.4.** Let $G$ be a graph. If $\varphi(G) \geq t$, then $G$ contains an induced minimal b-$t$-atom.

**Proof.** Suppose that $\varphi(G) = t'$, with $t' \geq t$. Thus, there exists a b-coloring of $G$ with $t'$ colors. Let $u_1, \ldots, u_{t'}$ be a set of b-vertices, each in a different color class of $V(G)$. The graph induced by $N[u_1] \cup \ldots \cup N[u_{t'}]$ contains a b-$t'$-atom. Hence, by Lemma 3.2, since $G$ contains an induced b-$t'$-atom, then it also contains an induced minimal b-$t$-atom.

**Theorem 3.5.** For a graph $G$, we have $\varphi_r(G) \geq t$ if and only if $G$ contains an induced minimal b-$t$-atom.

**Proof.** Suppose that the graph $G$ contains an induced b-$t$-atom $A$. Since $A$ admits, by definition, a b-$t$-coloring, we have $\varphi_r(G) \geq t$. Using Proposition 3.4, we obtain the converse.

**Definition 3.2.** Let $G$ be a graph. For an induced subgraph $A$ of $G$, let $N(A) = \{v \in V(G) \setminus V(A) | uv \in E(G), u \in V(A)\}$. A b-$t$-atom $A$ is feasible in $G$ if there exists a b-$t$-coloring of $V(A)$ that can be extended to the vertices of $N(A)$ without using new colors.

**Proposition 3.6.** Let $G$ be a graph. If $G$ contains an induced feasible minimal b-$t$-atom and no induced feasible minimal b-$t'$-atom, for $t' > t$, then $\varphi(G) = t$.

**Proof.** Suppose that $G$ contains an induced feasible minimal b-$t$-atom $A$ and no b-$t$-coloring of $G$ exists. We begin by considering that the vertices of $A \cup N(A)$ are already colored with $t$ colors. We can note that, by assumption, no coloring of $A \cup N(A)$ (from the definition) can be extended to the whole graph using only $t$ colors. Let $t'$ be the largest integer such that the coloring can not be extended.
to a $b$-$t'$-coloring of the whole graph and let $v$ be a vertex that can not be given a color among $\{1, \ldots, t'\}$. Thus, we suppose that the coloring can be extended to a $b$-$(t' + 1)$-coloring where $v$ is colored by $t' + 1$. Since $A \cup N(A)$ is already colored, we have $v \in V(G) \setminus (A \cup N(A))$. The vertex $v$ should be adjacent to vertices of every color, otherwise it could be colored. One vertex of each color class in $N(v)$ should be adjacent to vertices of each color class (except its color). Otherwise, the colors of the vertices of $N(v)$ could be changed in order that some color $c$ no longer appear in $N(v)$, and consequently $v$ can be recolored with color $c$. Then, the graph induced by the vertices at distance at most 2 from $v$ contains a $b$-$(t' + 1)$-atom where $N[v]$ contains the center of this $b$-$(t' + 1)$-atom. Moreover, this $b$-$(t' + 1)$-atom is feasible as the whole graph is $b$-$(t' + 1)$-colorable, contradicting the hypothesis.

**Proposition 3.7.** Let $G$ be a graph. If $\varphi(G) = t$, then $G$ contains an induced feasible minimal $b$-$t$-atom and no induced feasible minimal $b$-$t'$-atom, for $t' > t$.

**Proof.** Suppose $\varphi(G) = t$. By Proposition 3.4, $G$ contains an induced minimal $b$-$t$-atom. If no induced minimal $b$-$t$-atom is feasible, then there exists no $b$-$t$-coloring of $G$, a contradiction.

A direct consequence of Proposition 3.6 and Proposition 3.7 is the following.

**Theorem 3.8.** For a graph $G$, we have $\varphi(G) = t$ if and only if $G$ contains an induced feasible minimal $b$-$t$-atom and no induced feasible minimal $b$-$t'$-atom, for $t' > t$.

The following proposition will be useful in the last section.

**Proposition 3.9.** Let $G$ be a graph and let $t = \varphi_r(G)$. If every minimal $b$-$t$-atom is feasible in $G$, then $\varphi(G) = \varphi_r(G)$.

**Proof.** Since $t = \varphi_r(G)$, $G$ does not contain a $b$-$(t + 1)$-atom. Thus, by Proposition 3.6, we obtain $\varphi(G) = t$.

Note that the problem of determining if a graph has a $b$-$t$-coloring is NP-complete even if $t$ is fixed [21]. However, it does not imply that determining if $\varphi(G) \geq t$ for a graph $G$ is NP-complete. In contrast with the $b$-chromatic number, determining if a graph has $b$-relaxed number at least $t$ is in XP.

**Proposition 3.10.** Let $G$ be a graph of order $n$ and let $t$ be an integer. There exists an algorithm in time $O(n^{t^2})$ to determine if $\varphi_r(G) \geq t$. In particular, the problem $b$-$r$-$COL$ with parameter $t$ is in XP.

**Proof.** By Theorem 3.5, it suffices to verify that $G$ contains an induced minimal $b$-$t$-atom to determine if $\varphi_r(G) \geq t$. By Observation 3.1, the order of a minimal $b$-$t$-atom is bounded by $t^2$. Thus, we obtain an algorithm in time $O(n^{t^2})$.

Another NP-complete problem is to determine the $b$-spectrum of a graph $G$ [2], i.e. the set of integers $k$ such that $G$ is $b$-$k$-colorable. For a graph $G$ satisfying $\varphi(G) = \varphi_r(G)$, our algorithm can be used. Thus, proving that for a class of graphs, every graph $G$ satisfies $\varphi(G) = \varphi_r(G)$, implies that the problem $b$-$COL$ with parameter $t$ is in XP for this class of graphs.
4 b-critical vertices and edges

The concept of *b-critical vertices* and *b-critical edges* has been introduced recently and since five years a large number of articles are considering this subject \([1, 4, 5, 9, 24]\). In this section, we illustrate how this notion is strongly connected with the concept of b-\(t\)-atom.

**Definition 4.1** ([4, 9]). Let \(G\) be a graph. A vertex \(v\) of \(G\) is b-critical if \(\varphi(G - v) < \varphi(G)\). An edge \(e\) is b-critical if \(\varphi(G - e) < \varphi(G)\). A vertex \(v\) (edge \(e\), respectively) in a graph \(G\) is a b-\(t\)-trap, if there exists a b-\(t\)-atom of \(G\) that becomes feasible by removing \(v\) (\(e\), respectively).

**Proposition 4.1.** Let \(G\) be a graph. A vertex \(v\) is b-critical if and only if it is in every feasible minimal b-\(\varphi(G)\)-atom and \(v\) is not a b-\(\varphi(G)\)-trap.

**Proof.** Let \(t = \varphi(G)\). First, if \(v\) is not in a feasible minimal b-\(t\)-atom, then \(\varphi(G - v) = t\) and \(v\) is not b-critical. If \(v\) is a b-\(t\)-trap, then, by definition, \(\varphi(G - v) = t\). Second, suppose \(v\) is not a b-\(t\)-trap. If \(v\) is in every feasible minimal b-\(t\)-atom, then, since every minimal b-\(t\)-atom in \(G\) does not contain any other feasible minimal b-\(t\)-atom as induced subgraph, \(G - v\) does not contain a feasible minimal b-\(t\)-atom. Thus, \(v\) is b-critical. \(\Box\)

**Corollary 4.2.** If a graph \(G\) contains two induced feasible minimal b-\(\varphi(G)\)-atoms with disjoint set of vertices, then it contains no b-critical vertex.

**Proposition 4.3.** Let \(G\) be a graph and \(v\) be a vertex of \(V(G)\). If \(\varphi(G - v) > \varphi(G)\), then \(G\) contains a minimal b-\(\varphi(G - v)\)-atom which is not feasible. If \(\varphi(G - v) < \varphi(G) - 1\), then \(G - v\) contains no feasible minimal b-\(t\)-atom, for \(\varphi(G - v) < t \leq \varphi(G)\).

**Proof.** Note that every b-\(t\)-atom contained in \(G - v\) is also contained in \(G\), for any integer \(t\). Thus, if \(\varphi(G - v) > \varphi(G)\), then \(G\) contains a b-\(\varphi(G - v)\)-trap and consequently a minimal b-\(\varphi(G - v)\)-atom which is not feasible. Moreover, if \(\varphi(G - v) < \varphi(G) - 1\) and \(G - v\) contains a feasible b-\(t\)-atom for \(\varphi(G - v) < t \leq \varphi(G)\), then \(\varphi(G - v) \geq t\).

In [1], Balakrishnan and Raj have proved the following theorem.

**Theorem 4.4** ([1]). Let \(G\) be a graph and \(v\) be a vertex of \(V(G)\). We have \(\varphi(G) - \left\lfloor \frac{|V(G)|}{2} \right\rfloor + 2 \leq \varphi(G - v) \leq \varphi(G) + \left\lfloor \frac{|V(G)|}{2} \right\rfloor - 2\).

Moreover, they have determined the families of graphs for which there exists a vertex \(v\) such that \(\varphi(G - v) = \varphi(G) - \left\lfloor \frac{|V(G)|}{2} \right\rfloor + 2\) or \(\varphi(G - v) = \varphi(G) + \left\lfloor \frac{|V(G)|}{2} \right\rfloor - 2\). In contrast with the b-chromatic number, we have the following property about the b-relaxed number.

**Proposition 4.5.** Let \(G\) be a graph. If a vertex \(v\) is b-critical, then \(\varphi_r(G - v) = \varphi_r(G) - 1\).
Case 3:\( \varphi \) disjoint edges.\( v \) vertex \( \varphi \) Thus, we can suppose that \( \varphi \) are the only minimal b-4-atoms and b-5-atoms that contain at most five vertices \( K \) mal b-3-atoms that contain at most five vertices are \( K \)mal b-3-atoms that contains at most five vertices.\( K \)mal b-3-atoms that contains at most five vertices are.

Case 1: \( P \) contains any induced minimal b-3-atom and contain two disjoint edges.

Theorem 4.7. Let \( G \) be a graph. We have \( \varphi_r(G - v) = \varphi_r(G) + \lfloor \frac{|V(G)|}{2} \rfloor - 2 \), for every vertex \( v \) of \( V(G) \), if and only if \( G \) contains two disjoint edges but no induced minimal b-3-atom.

Proof. We can note that we have \( \varphi_r(G - v) = \varphi_r(G) + \lfloor \frac{|V(G)|}{2} \rfloor - 2 \) if and only if \( \varphi_r(G - v) = \varphi_r(G) \).

First, if \( G \) contains no minimal b-3-atom and contains an edge, then \( \varphi_r(G) = 2 \). Moreover, if \( G \) contains two disjoint edges, then for any vertex \( v \), \( G - v \) contains \( P_2 \) and \( \varphi_r(G - v) = 2 \).

Second, suppose that for every vertex \( v \), \( \varphi_r(G - v) = \varphi_r(G) \). The only minimal b-3-atoms that contains at most five vertices are \( K_3 \), \( C_5 \) and \( P_5 \). Moreover, the only minimal b-4-atoms and b-5-atoms that contain at most five vertices are \( K_4 \) and \( K_5 \). We are going to show that \( G \) is not one of these graphs.

Case 1: \( \varphi_r(G) = 5 \). If \( G \) is a \( K_5 \), then, by removing any vertex \( v \), we obtain \( \varphi_r(G - v) = 4 \).

Case 2: \( \varphi_r(G) = 4 \). If \( G \) is a \( K_4 \), then, by removing any vertex \( v \), we obtain \( \varphi_r(G - v) = 3 \). If \( G \) contains an induced \( K_4 \), \( |V(G)| = 5 \) and \( G \) is not \( K_5 \), then there exists a vertex \( v \) such \( G - v \) has no induced \( K_4 \) and \( \varphi_r(G - v) = 3 \).

Case 3: \( \varphi_r(G) = 3 \). If \( G \) contains an induced \( K_3 \) and no induced \( K_4 \), then, since the induced \( K_3 \) in \( G \) have a common vertex \( v \), we obtain \( \varphi_r(G - v) = 2 \). Moreover, if \( G \) is \( P_5 \) or \( C_5 \), then, by removing any vertex \( v \), we obtain \( \varphi_r(G - v) = 2 \).

Thus, we can suppose that \( \varphi_r(G) = 2 \). If \( G \) contains only edges with a common vertex \( v \), then \( \varphi_r(G - v) = 1 \). Hence, \( G \) contains no b-3-atom and contain two disjoint edges.

The following theorem is a generalization of a conjecture of Blidia et al. [3] for the parameter \( \varphi_r \). Note that the graphs \( P_4 \), \( C_4 \) and \( P_2 \cup P_2 \) do not contain any induced minimal b-3-atom and contain two disjoint edges.

Theorem 4.7. Let \( G \) be a graph. We have \( \varphi_r(G - v) = \varphi_r(G) + \lfloor \frac{|V(G)|}{2} \rfloor - 2 \), for every vertex \( v \) of \( V(G) \), if and only one of these conditions is true about \( G \):

i) \( G \) is \( P_2 \) or \( C_3 \).

ii) \( E(G) = \emptyset \) and \( 4 \leq |V(G)| \leq 5 \).

iii) \( 4 \leq |V(G)| \leq 5 \) and \( G \) contains two disjoint edges but no b-3-atom.
Proof. Note that if $|V(G)| \geq 6$, then, by Proposition 4.5, we can not have $\varphi_r(G - v) = \varphi_r(G) + \left\lfloor \frac{|V(G)|}{2} \right\rfloor - 2$. Note also that if $G$ contains only one vertex, then it can not satisfy $\varphi_r(G - v) = \varphi_r(G) + \left\lfloor \frac{|V(G)|}{2} \right\rfloor - 2$.

First, if $2 \leq |V(G)| \leq 3$, then we have $\varphi_r(G - v) = \varphi_r(G) - 1$ if and only if $G$ is a minimal b-$t$-atom. Hence, if and only if $G$ is $P_2$ or $C_3$. Second, if $G$ contains no edges, then $\varphi_r(G) = 1$ and for any vertex $v$, $\varphi_r(G - v) = 1$. The third condition is obtained by Lemma 4.6.

**Definition 4.2.** Let $t$ be a positive integer and $A$ be a b-$t$-atom. An edge $e$ is b-atom-critical in $A$ if $A - e$ is not a b-$t$-atom.

**Proposition 4.8.** Let $G$ be a graph. An edge $e$ is b-critical if and only if it is b-atom-critical in every feasible minimal b-$\varphi(G)$-atom and $e$ is not a b-$\varphi(G)$-trap.

**Proof.** Let $t = \varphi(G)$. First, if $e$ is not b-atom-critical in a feasible minimal b-$t$-atom, then $G - e$ contains a feasible minimal b-$t$-atom and $\varphi(G - e) = t$. If $e$ is a b-$t$-trap, then, by definition, $\varphi(G - e) = t$. Second, suppose that $e$ is not a b-$t$-trap. If $e$ is b-atom-critical in every feasible minimal b-$t$-atom, then, since every feasible minimal b-$t$-atom in $G$ does not contain any other feasible minimal b-$t$-atom as subgraph in $G - e$, the graph $G - e$ does not contain a feasible minimal b-$t$-atom. Thus, $e$ is b-critical.

**Corollary 4.9.** If a graph $G$ contains two induced feasible minimal b-$\varphi(G)$-atoms with disjoint sets of b-atom-critical edges, then $G$ contains no b-critical edge.

## 5 b-perfect graphs

A b-perfect graph is a graph for which every induced subgraph satisfies that its b-chromatic number is equal to its chromatic number. More generally, we present the following definitions.

**Definition 5.1** ([13]). A graph $G$ is b-$\chi$-$k$-bounded, for $k$ a positive integer, if $\varphi(G') - \chi(G') \leq k$, for every induced subgraph $G'$ of $G$. A graph $G$ is a $\chi$-$k$-unbounded b-atom, for $k$ a positive integer, if $\varphi(G) - \chi(G) > k$ and $G$ is a b-$t$-atom for some integer $t$. A graph $G$ is an imperfect b-atom, for $k$ a positive integer, if $\varphi(G) > \chi(G)$ and $G$ is a b-$t$-atom for some integer $t$.

Hoang et al. [14] characterized b-perfect graphs by giving the family $\mathcal{F}$ of forbidden induced subgraphs depicted in Figure 4. We recall the following theorem:

**Theorem 5.1** ([14]). A graph is b-perfect if and only if it contains no graph from $\mathcal{F}$ as induced subgraph.

Note that every graph in the family $\mathcal{F}$ is a b-$t$-atom for some $t$. More precisely, $F_1, F_2$ and $F_3$ are the only minimal bipartite b-3-atoms. The remaining
graphs are minimal b-4-atoms that do not contain $F_1$, $F_2$ and $F_3$ as induced subgraph and which admit a proper coloring with three colors (as mentioned in [15]). We can state the following property about b-4-atoms.

**Theorem 5.2.** Let $k$ be a positive integer. A graph $G$ is not $b$-$\chi$-$k$-bounded if and only if it contains a minimal $\chi$-$k$-unbounded b-atom.

**Proof.** First, if $G$ contains a minimal $\chi$-$k$-unbounded b-atom, then, by definition, $G$ is not $\chi$-$k$-bounded.

Second, suppose $G$ is not $b$-$\chi$-$k$-bounded. Then, there exists an induced subgraph $A$ of $G$ of minimal order which is not $b$-$\chi$-$k$-bounded. By removing vertices of $A$ we can only decrease the chromatic number. Thus, by removing vertices we can obtain a $b$-$\varphi(A)$-atom which is $\chi$-$k$-unbounded.

**Corollary 5.3.** The graphs with $b$-chromatic number $t$ which are $b$-$\chi$-$k$-bounded, for fixed integers $k$ and $t$, can be defined by forbidding a finite family of induced subgraphs: the $\chi$-$k$-unbounded b-atoms. Hence, a graph $G$ is $b$-perfect if and only if it does not contain imperfect b-atoms.
Let $b$-$\chi$-BOUNDED be the following decision problem and let $k$ be an integer, with $0 \leq k < \varphi(G)$.

**b-$\chi$-$k$-BOUNDED**

**Instance:** A graph $G$.

**Question:** Does $\varphi(G) - \chi(G) \geq k$?

By Corollary 5.3, we obtain the following corollary:

**Corollary 5.4.** Let $G$ be a graph and $k$ be an integer, with $0 \leq k < \varphi(G)$. There exists an algorithm in time $O(n^{\varphi(G)^2})$ to solve $b$-$\chi$-$k$-BOUNDED.

Since a graph $G$ is $b$-perfect if and only if it does not contain imperfect $b$-atoms, we have the following theorem:

**Theorem 5.5.** The number of imperfect $b$-atoms is finite. A graph is an imperfect $b$-atom if and only if it is in the family $F$ (Figure 4).

The previous theorem is a consequence of Theorem 5.1. Remark that if we can prove that every minimal $b$-$4$-atom except $K_4$ contains an induced subgraph of the family $F$, then, using Theorem 5.2, we obtain another proof of Theorem 5.1.

### 6 b-chromatic and b-relaxed chromatic numbers

In this section we consider the $b$-relaxed number relatively to the $b$-chromatic number and prove equality for trees and graphs of girth at least $7$.

**Lemma 6.1.** A minimal $b$-$t$-atom has at most $t$ connected components.

**Proof.** Suppose that a minimal $b$-$t$-atom $G$ has more than $t$ connected components. By definition, at least one connected component $A$ of $G$ does not contain a vertex of $C(G)$. Since $G - A$ is also a $b$-$t$-atom, $G$ is not minimal. \(\square\)

Note that a minimal $b$-$t$-atom $G$ contains a center $C(G)$ and the remaining vertices of $G$ are neighbors of vertices of $C(G)$.

**Proposition 6.2.** For a tree $T$, we have $\varphi(T) = \varphi_r(T)$.

**Proof.** Let $t = \varphi_r(T)$. By Proposition 3.9, it suffices to prove that every minimal $b$-$t$-atom is feasible to have $\varphi(T) = \varphi_r(T)$. Let $T'$ be a minimal $b$-$t$-atom and let $N[T'] = V(T') \cup N(T')$. By Lemma 6.1, $T'$ has at most $t$ connected components. Let $u$ be a vertex of $N(T')$ with a maximal number of neighbors in $N[T']$. Since $T'$ has at most $t$ connected components and $T$ is a tree, $u$ has at most $t$ neighbors in $N[T']$.

Our proof consists in extending the coloring of $T'$ induced by $D_1, \ldots, D_t$ to $N(T')$ using colors from $\{1, \ldots, t\}$. For $t = 2$, the proof is trivial since the only minimal $b$-$2$-atom is $P_2$ and we can easily extend the coloring to $N(P_2)$. Thus we can suppose that $t \geq 3$. If $u$ has at most $t - 1$ neighbors in $N[T']$,
then we can extend the coloring. Thus, we suppose that \( u \) has \( t \) neighbors in \( N[T'] \). In this case, \( T' \) has \( t \) connected components which are all stars. Each vertex of \( N(u) \cap N[T'] \) is either a vertex of a connected component of \( T' \) or a vertex in \( N(T') \) which is adjacent to one vertex of \( V(T') \). In these two cases the vertices of \( N(u) \cap N[T'] \) should be in or be adjacent to vertices of disjoint connected components of \( T' \). Thus the vertices of \( N(u) \cap N[T'] \) have at most two neighbors in \( N[T'] \): the vertex \( u \) and another vertex of \( T' \) (otherwise, there is a cycle in \( T' \)). We begin by giving a color from \( \{1, \ldots, t\} \) to the vertices of \( N(T') \setminus \{u\} \). The vertex \( u \) can not be adjacent to all vertices of \( C(T') \) since otherwise it would contradict \( t = \varphi_r(T) \). Let \( v \in N[T'] \setminus C(T') \) be a neighbor of \( u \). If \( v \in N(T') \), then \( v \) has at most two neighbors in \( N[T'] \) and \( v \) can be recolored in order to color \( u \). If all neighbors of \( u \) are in \( T' \), then \( v \in N(c_i) \), for \( i \in \{1, \ldots, t\} \) and we can exchange the color of \( v \) with the color of a vertex \( w \in N(c_i) \setminus \{v\} \) in order to color \( u \) (since \( t \geq 3 \), \( N(c_i) \setminus \{v\} \) is not empty). Finally, the vertices of \( N(u) \cap N(T') \) can be recolored if we have obtained an improper coloring by recoloring \( w \).

The girth of a graph \( G \) is the length of a smallest cycle in \( G \). We finish this paper by proving that when a graph \( G \) has sufficiently large girth, we have \( \varphi(G) = \varphi_r(G) \), thus extending Proposition 6.2.

**Theorem 6.3.** Let \( G \) be a graph with girth \( g \) and \( \varphi_r(G) \geq 3 \). If \( g \geq 7 \), then \( \varphi(G) = \varphi_r(G) \).

**Proof.** Let \( t = \varphi_r(G) \). By Proposition 3.9, it suffices to prove that every minimal b-t-atom is feasible to have \( \varphi(G) = \varphi_r(G) \). Let \( A_t \) be a minimal b-t-atom. Our proof consists in extending the coloring of \( A_t \) induced by \( D_1, \ldots, D_t \) to \( N(A_t) \) using colors from \( \{1, \ldots, t\} \). Thus, we consider that the vertices of \( A_t \) are already colored.

For a vertex \( u \in N(A_t) \), we denote by \( I_c(u) \) the set \( \{i \in \{1, \ldots, t\} \mid \exists v \in N(u) \cap N(c_i)\} \). For a vertex \( u \in V(A_t) \), we denote by \( c^u \) a neighbor of \( u \) in \( C(A_t) \) if \( u \notin C(A_t) \) or the vertex \( u \) itself if \( u \in C(A_t) \). Finally, we denote by \( N(A_t) \), the set of vertices \( V(A_t) \cup N(A_t) \). In the different cases, when we describe a cycle of length at most \( k \) by \( u_1, \ldots, u_k \), it is assumed that, depending the configuration, consecutive symbols can denote the same vertex. In this proof, any considered vertex is supposed to be in \( N(A_t) \). We begin by proving the following properties:

i) No vertex of \( N(A_t) \) is adjacent to two vertices of \( N(c_i) \), for \( 1 \leq i \leq t \);

ii) If \( u, v \in N(A_t) \) and \( i \in I_c(u) \cap I_c(v) \), then \( u \) and \( v \) are not adjacent and have no common neighbor in \( N(A_t) - c_i \);

iii) If \( u, v \in N(c_i) \) and \( u', v' \in N(c_j) \), \( u \neq v \), \( u' \neq v' \), for some \( i \) and \( j \), \( 1 \leq i < j \leq t \), then the subgraph induced by \( \{u, v, u', v'\} \) contains at most one edge.

i) If \( u \) is adjacent to two vertices of \( N(c_i) \), for some \( i \), \( 1 \leq i \leq t \), then \( u \) is in a cycle of length at most 4. This cycle contains \( u, c_i \) and one or two vertices of \( N[c_i] \).
ii) If $u$ and $v$ are adjacent or have a common neighbor, then $u$ and $v$ belong to a cycle of length at most 6. This cycle contains $u$, $v$, vertices of $N[c_i]$ and possibly the common neighbor of $u$ and $v$ in $N(A_t) - c_i$, for $i$ an integer such that $i \in I_c(u) \cap I_c(v)$.

iii) If the subgraph induced by $\{u, v, u', v'\}$ contains at least 2 edges, then there is a cycle of length at most 6 in $G$. This cycle is $u-v-c_i$ if $u$ and $v$ are adjacent, $u'-v'-c_j$ if $u'$ and $v'$ are adjacent or the cycle $u-c_i-v-u'-v'-c_j$, otherwise.

We are going to prove that either each vertex $u \in N(A_t)$ can be colored with colors from $\{1, \ldots, t\}$ or the graph $G$ contains a $b(t+1)$-atom (which contradicts $\varphi_c(G) = t$). By properties i) and ii), any vertex of $N(A_t)$ has at most $t$ neighbors in $N[A_t]$. Hence we may suppose that any vertex $u \in N(A_t)$ with less than $t$ neighbors in $N[A_t]$ is already colored and only consider vertices of $N(A_t)$ with $t$ neighbors in $N[A_t]$. For a vertex $u \in N[A_t]$, a color $i$ is said to be available for $u$ if no vertex has color $i$ in $N(u) \cap N[A_t]$ (and therefore, $u$ has no available color if the colors 1, $\ldots$, $t$ are not available for $u$). Let $N_u(A_t)$ be the set of vertices in $N(A_t)$ with no available colors.

We define the following three sets:

- $N_1 = \{u \in N(A_t)| N(u) \cap (V(A_t) \setminus C(A_t)) \neq \emptyset, N(u) \cap N(A_t) = \emptyset\}$;
- $N_2 = \{u \in N(A_t)| N(u) \cap (V(A_t) \setminus C(A_t)) \neq \emptyset, N(u) \cap N(A_t) \neq \emptyset\}$;
- $N_3 = \{u \in N(A_t)| N(u) \cap (V(A_t) \setminus C(A_t)) = \emptyset\}$.

We can remark that $N_1 \cup N_2 \cup N_3 = N(A_t)$.

In the remainder of the proof we will first consider the vertices of $N_1$; secondly the vertices of $N_2$; and finally the vertices of $N_3$.

**Case 1:** vertices of $N_1$.

Let $u$ be a vertex of $N_1$. We recall that, by the above assumption, $u$ has exactly $t$ neighbors in $A_t$. Moreover, by Property i), $|L_c(u)| = t$. Let $c_i \in C(A_t)$. We denote by $A^*_u$ the vertices of $N(c_i)$ which have a neighbor in $N_u(A_t)$. Notice that a vertex $v \in A^*_u$ can not have a neighbor $x$ in $V(A_t) \setminus \{c_i\}$ since otherwise it would create a cycle $v-x-c^*-v-u$, for $u$ the neighbor of $v$ in $N_1 \cap N_u(A_t)$ and $v'$ the neighbor of $u$ in $N[c^*]$. This cycle has length at most 5, contradicting $g \geq 7$. If for a vertex $c_i \in C(A_t)$ we have $|A^*_u| \geq 2$, we exchange the colors of the vertices of $A^*_u$ by doing a cyclic permutation of their colors. Afterwards, we obtain that some vertices of $N_1 \cap N_u(A_t)$ have now an available color and we recolor them by any available color. Finally, we color the vertices of $N_1$, when possible, by any available color. Let $N_u(A_t)$ be the set of the remaining uncolored vertices of $N_1$. In the following subcases, we recolor at most once the vertices of $N[c_i]$, for $i \in \{1, \ldots, t\}$, since any two vertices of $N_u(A_t)$ can not both have neighbors in $N(c_i)$. 

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Figure 5: Possible configurations in Subcases 1.2.1 (on the top) and 1.2.2 (on the bottom) before (on the left) and after (on the right) the recoloring process.

By considering that $N_*(A_t) \neq \emptyset$ (or else we have nothing more to do in Case 1)), we can suppose that for every two integers $i$, $j$, $1 \leq i \neq j \leq t$, we have $N[c_i] \cap N[c_j] = \emptyset$. Otherwise, if there exists a vertex $u \in N_*(A_t)$ and a vertex $w \in N[c_i] \cap N[c_j]$, there is a cycle $u-v-c_i-w-c_j-v'$ of length at most 6, for $v$ a neighbor of $u$ in $N[c_i]$ and $v'$ a neighbor of $u$ in $N[c_j]$.

Thus, we obtain that if $N_*(A_t) \neq \emptyset$, then every vertex $c_i \in C(A_t)$ has only one neighbor of color $j$, for $1 \leq i \neq j \leq t$, since otherwise it would contradict the minimality of $A_t$ (by removing one vertex of color $j$).

We then consider the two following subcases, for $u \in N_*(A_t)$.

**Subcase 1.1:** $u$ has exactly one neighbor in $V(A_t) \setminus C(A_t)$.

Let $v'$ be the neighbor of $u$ in $V(A_t) \setminus C(A_t)$ and let $c'$ be the color of $v'$.

Notice that no vertex $x$ from $N[c']$ has a neighbor $y$ in $V(A_t) \setminus N[c']$, since otherwise it would create a cycle $u-v'-x$-$y$-$c'$ of length at most 6. Consequently, we can exchange the color of $v'$ with the color of one vertex from $N(c')$ and color $u$ by $c'$.

**Subcase 1.2:** $u$ has more than one neighbor in $V(A_t) \setminus C(A_t)$.

Let $v_1$ and $v_2$ be two neighbors of $u$ in $V(A_t) \setminus C(A_t)$. Let $c'$ be the color of $v_1$ and let $c''$ be the color of $v_2$.

If $v_1$ has a neighbor $x \in V(A_t) \setminus N[c_1]$, then there exists a cycle $u-v_1-x-c''-v'$ in $G$, with $v'$ a neighbor of $c''$ in $N(u)$ (in the case $c''$ is not a neighbor of $u$). Similarly if $v_2$ has a neighbor in $V(A_t) \setminus N[c_2]$, then there is a cycle of length at most 5 in $G$. Consequently, we can suppose that $v_1$ has no neighbor in $V(A_t) \setminus N[c_1]$ and that $v_2$ has no neighbor in $V(A_t) \setminus N[c_2]$. If there exists a vertex of $N(c_1) \setminus \{v_1\}$ with no neighbor of color $c'$, then
we exchange the color of \( v_1 \) with the color of this vertex and color \( u \) by \( c' \). If there exists a vertex of \( N(c^{v_2}) \setminus \{v_2\} \) with no neighbor of color \( c'' \), then we exchange the color of \( v_2 \) with the color of this vertex and color \( u \) by \( c'' \). Thus, we may suppose that every vertex \( w \) of \( N(c^{v_1}) \setminus \{v_1\} \) (of \( N(c^{v_2}) \setminus \{v_2\} \), respectively) has a neighbor \( \overline{w} \) of color \( c'(c'') \), respectively) in \( V(A_t) \). We consider three subcases in order to color to \( u \).

**Subcase 1.2.1:** the vertices \( v_1 \) and \( c^{v_2} \) have the same color and the vertices \( v_2 \) and \( c^{v_1} \) have the same color.

Notice that no vertex \( w \in N(c^{v_1}) \) is adjacent to \( c^{v_2} \) since otherwise \( u \)-\( v_1 \)-\( c^{v_1} \)-\( w \)-\( c^{v_2} \)-\( v_2 \) would be a cycle of length at most 6 in \( G \). For the same reason, no vertex \( w \in N(c^{v_2}) \) is adjacent to \( c^{v_1} \). Thus, by Property iii), no vertex \( w \in N[c^{v_1}] \cup N[c^{v_2}] \) has a neighbor \( x \in V(A_t) \setminus (N[c^{v_1}] \cup N[c^{v_2}] \cup \{\overline{w}\}) \), since there exists a vertex \( y \in N(c^{w}) \) with neighbor \( \overline{y} \in N(c^{c''}) \). There could exist two adjacent vertices \( w \) and \( w' \) with \( w \in N(c^{v_1}) \) and \( w' \in N(c^{v_2}) \). However, the vertex \( w' \) has no neighbor of color \( c'' \) in \( A_t \) since \( w' \) and \( v_2 \) can not be adjacent and there does not exist a second vertex of color \( c'' \) in \( N(c^{v_2}) \). Consequently, we can exchange the color of \( v_1 \) with the color of \( v_2 \), the color of \( c^{v_1} \) with the color of \( c^{v_2} \) and afterward we can exchange the color of one vertex from \( N(c^{v_2}) \setminus \{v_1\} \) with the color of \( v_1 \) and color \( u \) by \( c'' \). The top of Figure 5 illustrates this recoloring process on a minimal b-4-atom fulfilling the hypothesis of Subcase 1.2.1.

**Subcase 1.2.2:** the vertices \( v_1 \) and \( c^{v_2} \) do not have the same color.

Let \( i \) be the color of \( c^{v_1} \) and \( j \) be the color of \( c^{v_2} \). In this case, we exchange the color of \( c^{v_1} \) with the color of \( c^{v_2} \) and the color of the vertex \( w \) of color \( j \) in \( N(c^{v_i}) \) with the color of the vertex \( w' \) of color \( i \) in \( N(c^{v_2}) \). For this, we have to suppose that \( w \) is not adjacent to a vertex of color \( i \) and that \( w' \) is not adjacent to a vertex of color \( j \). For \( t \geq 4 \), such vertices \( w \) and \( w' \) exist since at most one vertex of \( N(c^{v_1}) \) has a neighbor of color \( j \) (otherwise, it would contradict Property iii) since every vertex of \( N(c^{v_1}) \setminus \{v_1\} \) has already a neighbor in \( V(A_t) \) of color \( c' \) and at most one vertex of \( N(c^{v_2}) \) has a neighbor of color \( i \). If \( t = 3 \), then the only (up to isomorphism) b-3-atom with a coloring fulfilling all these hypothesis (up to color permutation) is illustrated at the bottom of Figure 5, along with the recoloring process. In this b-3-atom, no more edge can be added (otherwise, it would create a cycle of length at most 6).

**Subcase 1.2.3:** the vertices \( v_2 \) and \( c^{v_1} \) do not have the same color.

We proceed as for the previous subcase by considering \( v_2 \) instead of \( v_1 \) and \( c^{v_1} \) instead of \( c^{v_2} \).

**Case 2:** vertices of \( N_2 \).

Since each pair of adjacent vertices \( u, v \in N(A_t) \) satisfies Property ii), we obtain that \( I_c(u) \cap I_c(v) = \emptyset \). We color each vertex \( u \in N_2 \) by a color \( i \in I_c(u) \) such that \( u \) and \( c_i \) are not adjacent.
Case 3: vertices of $N_3$.
Notice that, by definition, a vertex of $C(A_t)$ has no available color. Let $u \in N_3$. We begin by coloring $u$ with any available color if it has some. If $u$ has no available color, there could exist a color $i$ such that every vertex of $N(u)$ with color $i$ has an available color (these vertices should be in $N(A_t)$). If such color $i$ exists, we recolor these vertices of color $i$ by any available color and give color $i$ to $u$. If such color $i$ does not exist, then the set of vertices at distance at most 2 from $u$ induces a $b-(t+1)$-atom with center $N[u]$. It can be noted that the recolored vertices are in $N(A_t)$ since $N(u) \cap V(A_t) \subseteq C(A_t)$.

We finish this proof by illustrating that the obtained coloring is a $b$-$t$-coloring of $N[A_t]$. In case 1, we have modified the coloring of $A_t$. However, since we have exchanged the colors of well-chosen vertices in order that every vertex of $C(A_t)$ still has neighbor of every color from $\{1, \ldots, t\}$ except its own color, this coloring remains a $b$-$t$-coloring. In case 3, we have only changed the color of vertices from $N(A_t)$.

We think that the previous theorem can be useful to determine the family of graphs of girth at least 7 satisfying $\varphi(G) = m(G)$. It has already been proven that graphs of girth at least 7 have $b$-chromatic number at least $m(G) - 1$ [7].

**Corollary 6.4.** Let $G$ be a graph of girth at least 7 and of order $n$ and let $t$ be an integer. There exists an algorithm in time $O(n^t^2)$ to determine if $\varphi(G) \geq t$.

## 7 Open questions

We conclude this article by listing few open questions.

1. For which family of graphs are the $b$-relaxed number and the $b$-chromatic number equal?

2. Does there exist an easy characterization of feasible $b$-$t$-atoms?

3. Does there exist an FPT algorithm, with parameter $t$, to determine if $\varphi(G) \geq t$?

## References


