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A characterization of b-chromatic and partial Grundy numbers by induced subgraphs

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Abstract

Gyárfás et al. and Zaker have proven that the Grundy number of a graph $G$ satisfies $\Gamma(G) \geq t$ if and only if $G$ contains an induced subgraph called a $t$-atom. The family of $t$-atoms has bounded order and contains a finite number of graphs. In this article, we introduce equivalents of $t$-atoms for b-coloring and partial Grundy coloring. This concept is used to prove that determining if $\varphi(G) \geq t$ and $\partial \Gamma(G) \geq t$ (under conditions for the b-coloring), for a graph $G$, is in XP with parameter $t$. We illustrate the utility of the concept of $t$-atoms by giving results on b-critical vertices and edges, on b-perfect graphs and on graphs of girth at least 7.

1 Introduction

Given a graph $G$, a proper $k$-coloring of $G$ is a surjective function $c : V(G) \rightarrow \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for any $uv \in E(G)$; the color class $V_i$ is the set $\{u \in V | c(u) = i\}$ and a vertex $v$ has color $i$ if $v \in V_i$. We denote by $N(u)$ the set of neighbors of a vertex $u$ and by $N[u]$ the set $N(u) \cup \{u\}$. A vertex $v$ of color $i$ is a Grundy vertex if it is adjacent to at least one vertex colored $j$, for every $j < i$. A Grundy $k$-coloring is a proper $k$-coloring such that every vertex is a Grundy vertex. The Grundy number of a graph $G$, denoted by $\Gamma(G)$, is the largest integer $k$ such that there exists a Grundy $k$-coloring of $G$ [10]. A partial Grundy $k$-coloring is a proper $k$-coloring such that every color class contains at least one Grundy vertex. The partial Grundy number of a graph $G$, denoted by $\partial \Gamma(G)$, is the largest integer $k$ such that there exists a partial Grundy $k$-coloring of $G$. Let $G$ and $G'$ be two graphs. By $G \cup G'$ we denote the graph with vertex set $V(G) \cup V(G')$ and edge set $E(G) \cup E(G')$. Let $m(G)$ be the largest integer
m such that $G$ has at least $m$ vertices of degree at least $m - 1$. A graph $G$ is tight if it has exactly $m(G)$ vertices of degree $m(G) - 1$.

Another coloring parameter with domination constraints on the colors is the \textit{b-chromatic number}. In a proper-$k$-coloring, a vertex $v$ of color $i$ is a $b$-vertex if $v$ is adjacent to at least one vertex colored $j$, $1 \leq j \neq i \leq k$. A $b$-$k$-coloring, also called b-coloring when $k$ is not specified, is a proper $k$-coloring such that every color class contains at least one $b$-vertex. The \textit{b-chromatic number} of a graph $G$, denoted by $\varphi(G)$, is the largest integer $k$ such that there exists a $b$-$k$-coloring of $G$. In this paper, we introduce the concept of \textit{b-relaxed number}, denoted by $\varphi_r(G)$. A \textit{b-$k$-relaxed coloring} of $G$ is a $b$-$k$-coloring of a subgraph of $G$. The $b$-relaxed number of $G$ is $\varphi_r(G) = \max_{H \subseteq G}(\varphi(H))$, for $H$ an induced subgraph of $G$. Note that we have $\varphi(G) \leq \varphi_r(G) \leq \partial\Gamma(G)$. The difference between $\varphi(G)$ and $\varphi_r(G)$ can be arbitrary large. Let $K_{n,n}^-$ denotes the complete bipartite graph $K_{n,n}$ in which we remove $n - 1$ pairwise non incident edges (or $n - 1$ edges of a perfect matching in $K_{n,n}$) [1]. For this graph we have $\varphi(K_{n,n}^-) = 2$ and $\varphi_r(K_{n,n}^-) = n$ as Figure 1 illustrates it (for $n = 3$).

The concept of b-coloring has been introduced by Irving and Manlove [16], and a large number of papers was published (see e.g. [8, 19]). The b-chromatic number of regular graphs has been investigated in a series of papers ([6, 17, 20, 22]). Determining the b-chromatic number of a tight graph is NP-hard even for a connected bipartite graph [18] and a tight chordal graph [12].

In this paper, we study the decision problems b-COL, b-r-COL and pG-COL with parameter $t$ from Table 1.

<table>
<thead>
<tr>
<th>Question</th>
<th>b-COL</th>
<th>b-r-COL</th>
<th>G-COL</th>
<th>pG-COL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complexity class</td>
<td>undetermined</td>
<td>XP</td>
<td>XP [23]</td>
<td>XP</td>
</tr>
</tbody>
</table>

Table 1: The different decision problems with input a graph $G$ and parameter $t$ and their complexity class.

A decision problem is in FPT with parameter $t$ if there exists an algorithm which resolves the problem in time $O(f(t) \cdot n^c)$, for an instance of size $n$, a computable
function $f$ and a constant $c$. A decision problem is in XP with parameter $t$ if there exists an algorithm which resolves the problem in time $O(f(t) n^{g(t)})$, for an instance of size $n$ and two computable functions $f$ and $g$.

The concept of $t$-atom was introduced independently by Gyárfás et al. [11] and by Zaker [23]. The family of $t$-atoms is finite and the presence of a $t$-atom can be determined in polynomial time for a fixed $t$. The following definition is slightly different from the definitions of Gyárfás et al. or Zaker, insisting more on the construction of every $t$-atom (some $t$-atoms cannot be obtained with the initial construction of Zaker).

Definition 1.1 ([23]). The family of $t$-atoms is denoted by $A^t$, for $t \geq 1$, and is defined by induction. The family $A^1$ only contains $K_1$. A graph $G$ is in $A^t$ if there exists a graph $G'$ in $A^t$ and an integer $m$, $m \leq |V(G')|$, such that $G$ is composed of $G'$ and an independent set $I_m$ of order $m$, adding edges between $G'$ and $I_m$ such that every vertex in $G'$ is adjacent to at least one vertex in $I_m$.

Moreover, in the following sections, we say that a graph $G$ in a family of graphs $F$ is minimal, if no graphs of $F$ is a proper induced subgraph of $G$. For example, a minimal $t$-atom $A$ is a $t$-atom for which there are no $t$-atoms which are induced in $A$ other than itself.

Theorem 1.1 ([11, 23]). A graph $G$ satisfies $\Gamma(G) \geq t$ if and only if it contains an induced minimal $t$-atom.

In this paper we prove equivalent theorems for $b$-relaxed number and partial Grundy number. In contrast with the minimal $t$-atoms, we can not define the minimal $t$-atoms for $b$-coloring as the smallest graphs such that $G$ satisfies $\varphi(G) = t$ (also called $b$-critical graphs).

The paper is organized as follows: Section 2 is devoted to the definition of $t$-atoms for the partial Grundy coloring. This concept allows us to prove that the partial Grundy coloring problem is in XP with parameter $t$. Section 3 is similar to Section 2 but for $b$-relaxed-coloring. Section 4 is devoted to the concept of $b$-critical vertices and edges. Section 5 is about $b$-perfect graphs. Finally, Section 6 deals with graphs for which the $b$-relaxed and the $b$-chromatic numbers are equal.

2 Partial-Grundy-$t$-atoms: $t$-atoms for partial Grundy coloring

We start this section with the definition of $t$-atoms for partial Grundy coloring.

Definition 2.1. Given an integer $t$, a partial Grundy $t$-atom (or pG-$t$-atom, for short) is a graph $A$ whose vertex-set can be partitioned into $t$ sets $D_1, \ldots, D_t$, where $D_i$ contains a special vertex $c_i$ for each $i \in \{1, \ldots, t\}$ such that the following holds:

- For all $i \in \{1, \ldots, t\}$, $D_i$ is an independent set and $|D_i| \leq t - i + 1$;
Figure 2: The minimal pG-2-atom (on the left) and the three minimal pG-3-atoms (the numbers are the colors of the vertices and the surrounded vertices form the centers).

- For all $i \in \{2, \ldots, t\}$, $c_i$ has a neighbor in each of $D_1, \ldots, D_{i-1}$.

The set $\{c_1, \ldots, c_t\}$ is called the center of $A$ and denoted by $C(A)$.

Note that the sets $D_1, \ldots, D_t$ induce a partial Grundy coloring of the pG-$t$-atom. Figure 2 illustrates several pG-$t$-atoms (and their induced colorings) obtained using the previous definition.

Observation 2.1. For every pG-$t$-atom $G$, we have $|V(G)| \leq \frac{(t+1)^2}{2}$.

Lemma 2.2. Let $t$ and $t'$ be two integers such that $1 \leq t' < t$. Every pG-$t$-atom contains a pG-$t'$-atom as induced subgraph.

Proof. Every pG-$t$-atom $G$ contains a pG-$t'$-atom $G'$: we can obtain $G'$ by removing every vertex in $D_k$, for $t' < k \leq t$, and by removing, afterwards, the vertices of $G'$ not adjacent to any vertex in $\{c_1, \ldots, c_{t'}\}$.

Note that the only minimal pG-2-atom is $P_2$. The minimal pG-3-atoms are $C_3$, $P_4$ and $P_2 \cup P_2$, $P_3$, $P_4$. These graphs are illustrated in Figure 2.

Theorem 2.3. For a graph $G$, we have $\partial \Gamma(G) \geq t$ if and only if $G$ contains an induced minimal pG-$t$-atom.

Proof. Suppose that $\partial \Gamma(G) = t' \geq t$. By definition, there exists a partial Grundy coloring of $G$ with $t'$ colors. Let $u_1, \ldots, u_{t'}$ be a set of Grundy vertices, each in a different color class of $V(G)$. The graph induced by $N[u_1] \cup \ldots \cup N[u_{t'}]$ contains a pG-$t'$-atom. Hence, by Lemma 2.2, since $G$ contains an induced pG-$t'$-atom, then it also contains an induced minimal pG-$t$-atom.

Suppose $G$ contains an induced minimal pG-$t$-atom. Thus, the sets $D_1, \ldots, D_t$ induce a partial-Grundy coloring of this pG-$t$-atom. We can extend this coloring to a partial Grundy coloring of $G$ with at least $t$ colors in a greedy way by coloring the remaining vertices in any order, assigning to each of them the smallest color not used by its neighbors.
Proposition 2.4. Let $G$ be a graph of order $n$ and let $t$ be an integer. There exists an algorithm in time $O(n \frac{t(t+1)}{2})$ to determine if $\partial \Gamma(G) \geq t$. Hence, the problem $pG$-COL with parameter $t$ is in XP.

Proof. By Theorem 2.3, it suffices to verify that $G$ contains an induced minimal $pG$-$t$-atom to have $\partial \Gamma(G) \geq t$. Since the order of a minimal $pG$-$t$-atom is bounded by $\frac{t(t+1)}{2}$, we obtain an algorithm in time $O(n \frac{t(t+1)}{2})$.

We finish this section by determining every graph $G$ with $\partial \Gamma(G) = 2$.

Proposition 2.5. For a graph $G$ without isolated vertices, we have $\partial \Gamma(G) = 2$ if and only if $G = K_{n,m}$, for $n \geq 2$ and $m \geq 1$ or $G$ only contains isolated edges.

Proof. Zaker [23] has proven that $\Gamma(G) = 2$ if and only if $G$ is the disjoint union of copies of some $K_{n,m}$, for $n \geq 1$ and $m \geq 1$. Let $n$ and $m$ be positive integers. We can note that a graph containing a copy of $K_{n,m}$, for $n \geq 2$ and $m \geq 1$ and a copy of $K_{n,m}$, for $n \geq 1$ and $m \geq 1$ contains an induced $P_3 \cup P_2$, hence a $pG$-3-atom. Hence, if $\partial \Gamma(G) = 2$, then $G = K_{n,m}$, for $n \geq 2$ and $m \geq 1$ or $G$ only contains isolated edges.

Moreover, neither $K_{n,m}$ nor $P_2 \cup \ldots \cup P_2$ does contain an induced $C_3$, $P_4$ or $P_3 \cup P_2$. Hence, $\partial \Gamma(K_{n,m}) = 2$.

3 b-t-atoms: t-atoms for b-coloring

As in the previous section, we start this section with the definition of b-t-atoms (the notion of t-atom for b-coloring).

Definition 3.1. Given an integer $t$, a b-t-atom is a graph $A$ whose vertex-set can be partitioned into $t$ sets $D_1, \ldots, D_t$, where $D_i$ contains a special vertex $c_i$ for each $i \in \{1, \ldots, t\}$ such that the following holds:

- For each $i \in \{1, \ldots, t\}$, $D_i$ is an independent set and $|D_i| \leq t$;
- For all $i, j \in \{1, \ldots, t\}$, with $i \neq j$, $c_i$ has a neighbor in $D_j$.

The set $\{c_1, \ldots, c_t\}$ is called the center of $A$ and denoted by $C(A)$.

Note that the sets $D_1, \ldots, D_t$ induce a b-coloring of the b-t-atom. Figure 3 illustrates several b-t-atoms (and their induced coloring) obtained using the previous definition.

Observation 3.1. For every b-t-atom $G$, we have $|V(G)| \leq t^2$.

Lemma 3.2. Let $t$ and $t'$ be two integers such that $1 \leq t' < t$. Every b-t-atom contains a b-$t'$-atom as induced subgraph.

Proof. Every b-t-atom $G$ contains a b-$t'$-atom $G'$: we can obtain $G'$ by removing every vertex in $D_k$, for $t' < k \leq t$, and by removing, afterwards, the vertices not adjacent to any vertex in $\{c_1, \ldots, c_{t'}\}$.


Figure 3: The minimal b-2-atom (on the left) and the five minimal b-3-atoms.

Note that the only minimal b-2-atom is $P_2$. The minimal b-3-atoms are $C_3$, $P_5$, $C_5$, $P_3 \cup P_4$ and $P_3 \cup P_3 \cup P_3$. These graphs are illustrated in Figure 3.

**Observation 3.3.** Every minimal pG-t-atom is an induced subgraph of a minimal b-t-atom or a minimal t-atom (an atom for the Grundy number).

**Proposition 3.4.** Let $G$ be a graph. If $\varphi(G) \geq t$, then $G$ contains an induced minimal b-t-atom.

**Proof.** Suppose that $\varphi(G) = t'$, with $t' \geq t$. Thus, there exists a b-coloring of $G$ with $t'$ colors. Let $u_1, \ldots, u_{t'}$ be a set of b-vertices, each in a different color class of $V(G)$. The graph induced by $N[u_1] \cup \ldots \cup N[u_{t'}]$ contains a b-$t'$-atom. Hence, by Lemma 3.2, since $G$ contains an induced b-$t'$-atom, then it also contains an induced minimal b-t-atom.

**Theorem 3.5.** For a graph $G$, we have $\varphi_r(G) \geq t$ if and only if $G$ contains an induced minimal b-t-atom.

**Proof.** Suppose that the graph $G$ contains an induced b-t-atom $A$. Since $A$ admits, by definition, a b-t-coloring, we have $\varphi_r(G) \geq t$. Using Proposition 3.4, we obtain the converse.

**Definition 3.2.** Let $G$ be a graph. For an induced subgraph $A$ of $G$, let $N(A) = \{v \in V(G) \setminus V(A) \mid uv \in E(G), \ u \in V(A)\}$. A b-t-atom $A$ is feasible in $G$ if there exists a b-t-coloring of $V(A)$ that can be extended to the vertices of $N(A)$ without using new colors.

**Proposition 3.6.** Let $G$ be a graph. If $G$ contains an induced feasible minimal b-t-atom and no induced feasible minimal b-$t'$-atom, for $t' > t$, then $\varphi(G) = t$.

**Proof.** Suppose that $G$ contains an induced feasible minimal b-t-atom $A$ and no b-t-coloring of $G$ exists. We begin by considering that the vertices of $A \cup N(A)$ are already colored with $t$ colors. We can note that, by assumption, no coloring of $A \cup N(A)$ (from the definition) can be extended to the whole graph using only $t$ colors. Let $t'$ be the largest integer such that the coloring can not be extended
Proof. By Theorem 3.5, it suffices to verify that problem b-r-COL with parameter $t$ exists an algorithm in time $O(t^n)$ determining if a graph has a b-relaxed number at least $\phi(G)$ if and only if $\phi(G) \geq t$. The vertex $v$ should be adjacent to vertices of every color, otherwise it could be colored. One vertex of each color class in $N(v)$ should be adjacent to vertices of each color class (except its color). Otherwise, the colors of the vertices of $N(v)$ could be changed in order that some color $c$ no longer appear in $N(v)$, and consequently $v$ can be recolored with color $c$. Then, the graph induced by the vertices at distance at most 2 from $v$ contains a b-($t' + 1$)-atom where $N[v]$ contains the center of this b-($t' + 1$)-atom. Moreover, this b-($t' + 1$)-atom is feasible as the whole graph is $b$-($t' + 1$)-colorable, contradicting the hypothesis.

Proposition 3.7. Let $G$ be a graph. If $\phi(G) = t$, then $G$ contains an induced feasible minimal b-t-atom and no induced feasible minimal b-t'-atom, for $t' > t$.

Proof. Suppose $\phi(G) = t$. By Proposition 3.4, $G$ contains an induced minimal b-t-atom. If no induced minimal b-t-atom is feasible, then there exists no b-t-coloring of $G$, a contradiction.

A direct consequence of Proposition 3.6 and Proposition 3.7 is the following.

Theorem 3.8. For a graph $G$, we have $\phi(G) = t$ if and only if $G$ contains an induced feasible minimal b-t-atom and no induced feasible minimal b-t'-atom, for $t' > t$.

The following proposition will be useful in the last section.

Proposition 3.9. Let $G$ be a graph and let $t = \phi_r(G)$. If every minimal b-t-atom is feasible in $G$, then $\phi(G) = \phi_r(G)$.

Proof. Since $t = \phi_r(G)$, $G$ does not contain a b-($t + 1$)-atom. Thus, by Proposition 3.8, we obtain $\phi(G) = t$.

Note that the problem of determining if a graph has a b-t-coloring is NP-complete even if $t$ is fixed [21]. However, it does not imply that determining if $\phi(G) \geq t$ for a graph $G$ is NP-complete. In contrast with the b-chromatic number, determining if a graph has a-r-relaxed number at least $t$ is in XP.

Proposition 3.10. Let $G$ be a graph of order $n$ and let $t$ be an integer. There exists an algorithm in time $O(n^{t^2})$ to determine if $\phi_r(G) \geq t$. In particular, the problem b-r-COL with parameter $t$ is in XP.

Proof. By Theorem 3.5, it suffices to verify that $G$ contains an induced minimal b-t-atom to determine if $\phi_r(G) \geq t$. By Observation 3.1, the order of a minimal b-t-atom is bounded by $t^2$. Thus, we obtain an algorithm in time $O(n^{t^2})$.

Another NP-complete problem is to determine the b-spectrum of a graph $G$ [2], i.e. the set of integers $k$ such that $G$ is b-k-colorable. For a graph $G$ satisfying $\phi(G) = \phi_r(G)$, our algorithm can be used. Thus, proving that for a class of graphs, every graph $G$ satisfies $\phi(G) = \phi_r(G)$, implies that the problem b-COL with parameter $t$ is in XP for this class of graphs.
4 b-critical vertices and edges

The concept of b-critical vertices and b-critical edges has been introduced recently and since five years a large number of articles are considering this subject [1, 4, 5, 9, 24]. In this section, we illustrate how this notion is strongly connected with the concept of b-t-atom.

Definition 4.1 ([4, 9]). Let G be a graph. A vertex v of G is b-critical if \( \phi(G - v) < \phi(G) \). An edge e is b-critical if \( \phi(G - e) < \phi(G) \). A vertex v (edge e, respectively) in a graph G is a b-t-trap, if there exists a b-t-atom of G that becomes feasible by removing v (e, respectively).

Proposition 4.1. Let G be a graph. A vertex v is b-critical if and only if it is in every feasible minimal b-\( \phi(G) \)-atom and v is not a b-\( \phi(G) \)-trap.

Proof. Let \( t = \phi(G) \). First, if \( v \) is not in a feasible minimal b-t-atom, then \( \phi(G - v) = t \) and v is not b-critical. If \( v \) is a b-t-trap, then, by definition, \( \phi(G - v) = t \). Second, suppose \( v \) is not a b-t-trap. If \( v \) is in every feasible minimal b-t-atom, then, since every minimal b-t-atom in G does not contain any other feasible minimal b-t-atom as induced subgraph, \( G - v \) does not contain a feasible minimal b-t-atom. Thus, \( v \) is b-critical.

Corollary 4.2. If a graph G contains two induced feasible minimal b-\( \phi(G) \)-atoms with disjoint set of vertices, then it contains no b-critical vertex.

Proposition 4.3. Let G be a graph and v be a vertex of V(G). If \( \phi(G - v) > \phi(G) \), then G contains a minimal b-\( \phi(G - v) \)-atom which is not feasible. If \( \phi(G - v) < \phi(G) - 1 \), then \( G - v \) contains no feasible minimal b-t-atom, for \( \phi(G - v) < t \leq \phi(G) \).

Proof. Note that every b-t-atom contained in \( G - v \) is also contained in G, for any integer t. Thus, if \( \phi(G - v) > \phi(G) \), then G contains a b-\( \phi(G - v) \)-trap and consequently a minimal b-\( \phi(G - v) \)-atom which is not feasible. Moreover, if \( \phi(G - v) < \phi(G) - 1 \) and \( G - v \) contains a feasible b-t-atom for \( \phi(G - v) < t \leq \phi(G) \), then \( \phi(G - v) \geq t \).

In [1], Balakrishnan and Raj have proved the following theorem.

Theorem 4.4 ([1]). Let G be a graph and v be a vertex of V(G). We have \( \phi(G) - [\frac{|V(G)|}{2}] + 2 \leq \phi(G - v) \leq \phi(G) + [\frac{|V(G)|}{2}] - 2 \).

Moreover, they have determined the families of graphs for which there exists a vertex v such that \( \phi(G - v) = \phi(G) - [\frac{|V(G)|}{2}] + 2 \) or \( \phi(G - v) = \phi(G) + [\frac{|V(G)|}{2}] - 2 \). In contrast with the b-chromatic number, we have the following property about the b-relaxed number.

Proposition 4.5. Let G be a graph. If a vertex v is b-critical, then \( \phi_r(G - v) = \phi_r(G) - 1 \). 

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Proof. By Proposition 4.1, \( v \) is in every \( b\varphi(G) \)-atom. Let \( i \) be the integer associated to \( v \) in the construction of this \( b\varphi(G) \)-atom. By removing the vertices with associated integer \( i \), we obtain a \( b(\varphi(G) - 1) \)-atom and thus \( \varphi_r(G - v) = \varphi_r(G) - 1 \).

Note that this proposition was already proved for trees [4].

Lemma 4.6. Let \( G \) be a graph with \( 4 \leq |V(G)| \leq 5 \) and \( E(G) \neq \emptyset \). We have \( \varphi_r(G - v) = \varphi_r(G) + \lfloor \frac{|V(G)|}{2} \rfloor - 2 \), for every vertex \( v \) of \( V(G) \), if and only if \( G \) contains two disjoint edges but no induced minimal \( b \)-3-atoms but no induced minimal \( b \)-3-atoms.

Proof. We can note that we have \( \varphi_r(G - v) = \varphi_r(G) + \lfloor \frac{|V(G)|}{2} \rfloor - 2 \) if and only if \( \varphi_r(G - v) = \varphi_r(G) \).

First, if \( G \) contains no minimal \( b \)-3-atoms and contains an edge, then \( \varphi_r(G) = 2 \). Moreover, if \( G \) contains two disjoint edges, then for any vertex \( v \), \( G - v \) contains \( G_2 \) and \( \varphi_r(G - v) = 2 \).

Second, suppose that for every vertex \( v \), \( \varphi_r(G - v) = \varphi_r(G) \). The only minimal \( b \)-3-atoms that contains at most five vertices are \( K_3 \), \( C_5 \) and \( P_5 \). Moreover, the only minimal \( b \)-4-atoms and \( b \)-5-atoms that contain at most five vertices are \( K_4 \) and \( K_5 \). We are going to show that \( G \) is not one of these graphs

Case 1: \( \varphi_r(G) = 5 \). If \( G \) is a \( K_5 \), then, by removing any vertex \( v \), we obtain \( \varphi_r(G - v) = 4 \).

Case 2: \( \varphi_r(G) = 4 \). If \( G \) is a \( K_4 \), then, by removing any vertex \( v \), we obtain \( \varphi_r(G - v) = 3 \). If \( G \) contains an induced \( K_4 \), \( |V(G)| = 5 \) and \( G \) is not \( K_5 \), then there is a vertex \( v \) such \( G - v \) has no induced \( K_4 \) and \( \varphi_r(G - v) = 3 \).

Case 3: \( \varphi_r(G) = 3 \). If \( G \) contains an induced \( K_3 \) and no induced \( K_4 \), then, since the induced \( K_3 \) in \( G \) has a common vertex \( v \), we obtain \( \varphi_r(G - v) = 2 \). Moreover, if \( G \) is \( P_3 \) or \( C_5 \), then, by removing any vertex \( v \), we obtain \( \varphi_r(G - v) = 2 \).

Thus, we can suppose that \( \varphi_r(G) = 2 \). If \( G \) contains only edges with a common vertex \( v \), then \( \varphi_r(G - v) = 1 \). Hence, \( G \) contains no \( b \)-3-atom and contains no induced \( b \)-3-atom and contain two disjoint edges.

The following theorem is a generalization of a conjecture of Blidia et al. [3] for the parameter \( \varphi_r \). Note that the graphs \( P_4 \), \( C_4 \) and \( P_2 \cup P_2 \) do not contain any induced minimal \( b \)-3-atom and contain two disjoint edges.

Theorem 4.7. Let \( G \) be a graph. We have \( \varphi_r(G - v) = \varphi_r(G) + \lfloor \frac{|V(G)|}{2} \rfloor - 2 \), for every vertex \( v \) of \( V(G) \), if and only one of these conditions is true about \( G \):

i) \( G \) is \( P_2 \) or \( C_3 \).

ii) \( E(G) = \emptyset \) and \( 4 \leq |V(G)| \leq 5 \).

iii) \( 4 \leq |V(G)| \leq 5 \) and \( G \) contains two disjoint edges but no \( b \)-3-atom.
Proof. Note that if $|V(G)| \geq 6$, then, by Proposition 4.5, we can not have $\varphi_r(G - v) = \varphi_r(G) + \lfloor |V(G)|/2 \rfloor - 2$. Note also that if $G$ contains only one vertex, then it can not satisfy $\varphi_r(G - v) = \varphi_r(G) + \lfloor |V(G)|/2 \rfloor - 2$.

First, if $2 \leq |V(G)| \leq 3$, then we have $\varphi_r(G - v) = \varphi_r(G) - 1$ if and only if $G$ is a minimal $b$-$t$-atom. Hence, if and only if $G$ is $P_2$ or $C_3$. Second, if $G$ contains no edges, then $\varphi_r(G) = 1$ and for any vertex $v$, $\varphi_r(G - v) = 1$. The third condition is obtained by Lemma 4.6.

Definition 4.2. Let $t$ be a positive integer and $A$ be a $b$-$t$-atom. An edge $e$ is $b$-atom-critical in $A$ if $A - e$ is not a $b$-$t$-atom.

Proposition 4.8. Let $G$ be a graph. An edge $e$ is $b$-critical if and only if it is $b$-atom-critical in every feasible minimal $b$-$\varphi(G)$-atom and $e$ is not a $b$-$\varphi(G)$-trap.

Proof. Let $t = \varphi(G)$. First, if $e$ is not $b$-atom-critical in a feasible minimal $b$-$t$-atom, then $G - e$ contains a feasible minimal $b$-$t$-atom and $\varphi(G - e) = t$. If $e$ is a $b$-$t$-trap, then, by definition, $\varphi(G - e) = t$. Second, suppose that $e$ is not a $b$-$t$-trap. If $e$ is $b$-atom-critical in every feasible minimal $b$-$t$-atom, then, since every feasible minimal $b$-$t$-atom in $G$ does not contain any other feasible minimal $b$-$t$-atom as subgraph in $G - e$, the graph $G - e$ does not contain a feasible minimal $b$-$t$-atom. Thus, $e$ is $b$-critical.

Corollary 4.9. If a graph $G$ contains two induced feasible minimal $b$-$\varphi(G)$-atoms with disjoint sets of $b$-atom-critical edges, then $G$ contains no $b$-critical edge.

5 b-perfect graphs

A b-perfect graph is a graph for which every induced subgraph satisfies that its $b$-chromatic number is equal to its chromatic number. More generally, we present the following definitions.

Definition 5.1 ([13]). A graph $G$ is $b$-$\chi$-$k$-bounded, for $k$ a positive integer, if $\varphi(G') - \chi(G') \leq k$, for every induced subgraph $G'$ of $G$. A graph $G$ is a $\chi$-$k$-unbounded b-atom, for $k$ a positive integer, if $\varphi(G) - \chi(G) > k$ and $G$ is a $b$-$t$-atom for some integer $t$. A graph $G$ is an imperfect $b$-atom, for $k$ a positive integer, if $\varphi(G) > \chi(G) + 1$ and $G$ is a $b$-$t$-atom for some integer $t$.

Hoang et al. [14] characterized b-perfect graphs by giving the family $F$ of forbidden induced subgraphs depicted in Figure 4. We recall the following theorem:

Theorem 5.1 ([14]). A graph is b-perfect if and only if it contains no graph from $F$ as induced subgraph.

Note that every graph in the family $F$ is a $b$-$t$-atom for some $t$. More precisely, $F_1$, $F_2$ and $F_3$ are the only minimal bipartite b-3-atoms. The remaining
graphs are minimal b-4-atoms that do not contain $F_1$, $F_2$ and $F_3$ as induced subgraph and which admit a proper coloring with three colors (as mentioned in [15]). We can state the following property about b-t-atoms.

Theorem 5.2. Let $k$ be a positive integer. A graph $G$ is not $b$-$\chi$-$k$-bounded if and only if it contains a minimal $\chi$-$k$-unbounded b-atom.

Proof. First, if $G$ contains a minimal $\chi$-$k$-unbounded b-atom, then, by definition, $G$ is not $\chi$-$k$-bounded.

Second, suppose $G$ is not $b$-$\chi$-$k$-bounded. Then, there exists an induced subgraph $A$ of $G$ of minimal order which is not $b$-$\chi$-$k$-bounded. By removing vertices of $A$ we can only decrease the chromatic number. Thus, by removing vertices we can obtain a $b$-$\varphi(A)$-atom which is $\chi$-$k$-unbounded. \qed

Corollary 5.3. The graphs with $b$-chromatic number $t$ which are $b$-$\chi$-$k$-bounded, for fixed integers $k$ and $t$, can be defined by forbidding a finite family of induced subgraphs: the $\chi$-$k$-unbounded b-atoms. Hence, a graph $G$ is $b$-perfect if and only if it does not contain imperfect b-atoms.
Let $b\chi$-BOUNDED be the following decision problem and let $k$ be an integer, with $0 \leq k < \varphi(G)$.

**$b\chi$-$k$-BOUNDED**

**Instance**: A graph $G$.

**Question**: Does $\varphi(G) - \chi(G) \geq k$?

By Corollary 5.3, we obtain the following corollary:

**Corollary 5.4.** Let $G$ be a graph and $k$ be an integer, with $0 \leq k < \varphi(G)$. There exists an algorithm in time $O(n^{2\varphi(G)^2})$ to solve $b\chi$-$k$-BOUNDED.

Since a graph $G$ is $b$-perfect if and only if it does not contain imperfect $b$-atoms, we have the following theorem:

**Theorem 5.5.** The number of imperfect $b$-atoms is finite. A graph is an imperfect $b$-atom if and only if it is in the family $F$ (Figure 4).

The previous theorem is a consequence of Theorem 5.1. Remark that if we can prove that every minimal $b$-4-atom except $K_4$ contains an induced subgraph of the family $F$, then, using Theorem 5.2, we obtain another proof of Theorem 5.1.

### 6 $b$-chromatic and $b$-relaxed chromatic numbers

In this section we consider the $b$-relaxed number relatively to the $b$-chromatic number and prove equality for trees and graphs of girth at least 7.

**Lemma 6.1.** A minimal $b$-$t$-atom has at most $t$ connected components.

**Proof.** Suppose that a minimal $b$-$t$-atom $G$ has more than $t$ connected component. By definition, at least one connected component $A$ of $G$ does not contain a vertex of $C(G)$. Since $G - A$ is also a $b$-$t$-atom, $G$ is not minimal.

Note that a minimal $b$-$t$-atom $G$ contains a center $C(G)$ and the remaining vertices of $G$ are neighbors of vertices of $C(G)$.

**Proposition 6.2.** For a tree $T$, we have $\varphi(T) = \varphi_r(T)$.

**Proof.** Let $t = \varphi_r(T)$. By Proposition 3.9, it suffices to prove that every minimal $b$-$t$-atom is feasible to have $\varphi(T) = \varphi_r(T)$. Let $T'$ be a minimal $b$-$t$-atom and let $N[T'] = V(T') \cup N(T')$. By Lemma 6.1, $T'$ has at most $t$ connected components. Let $u$ be a vertex of $N(T')$ with a maximal number of neighbors in $N[T']$. Since $T'$ has at most $t$ connected components and $T$ is a tree, $u$ has at most $t$ neighbors in $N[T']$.

Our proof consists in extending the coloring of $T'$ induced by $D_1, \ldots, D_t$ to $N(T')$ using colors from $\{1, \ldots, t\}$. For $t = 2$, the proof is trivial since the only minimal $b$-$2$-atom is $P_2$ and we can easily extend the coloring to $N(P_2)$.

Thus we can suppose that $t \geq 3$. If $u$ has at most $t - 1$ neighbors in $N[T']$,
then we can extend the coloring. Thus, we suppose that $u$ has $t$ neighbors in $N[T']$. In this case, $T'$ has $t$ connected components which are all stars. Each vertex of $N(u) \cap N[T']$ is either a vertex of a connected component of $T'$ or a vertex in $N(T')$ which is adjacent to one vertex of $V(T')$. In these two cases the vertices of $N(u) \cap N[T']$ should be in or be adjacent to vertices of disjoint connected components of $T'$. Thus the vertices of $N(u) \cap N(T')$ have at most two neighbors in $N[T']$: the vertex $u$ and another vertex of $T'$ (otherwise, there is a cycle in $T$). We begin by giving a color from $\{1, \ldots, t\}$ to the vertices of $N(T') \setminus \{u\}$. The vertex $u$ can not be adjacent to all vertices of $C(T')$ since otherwise it would contradict $t = \varphi_r(T)$. Let $v \in N[T'] \setminus C(T')$ be a neighbor of $u$. If $v \in N(T')$, then $v$ has at most two neighbors in $N[T']$ and $v$ can be recolored in order to color $u$. If all neighbors of $u$ are in $T'$, then $v \in N(c_i)$, for $i \in \{1, \ldots, t\}$ and we can exchange the color of $v$ with the color of a vertex $w \in N(c_i) \setminus \{v\}$ in order to color $u$ (since $t \geq 3$, $N(c_i) \setminus \{v\}$ is not empty). Finally, the vertices of $N(u) \cap N(T')$ can be recolored if we have obtained an improper coloring by recoloring $w$. 

The girth of a graph $G$ is the length of a smallest cycle in $G$. We finish this paper by proving that when a graph $G$ has sufficiently large girth, we have $\varphi(G) = \varphi_r(G)$, thus extending Proposition 6.2.

**Theorem 6.3.** Let $G$ be a graph with girth $g$ and $\varphi_r(G) \geq 3$. If $g \geq 7$, then $\varphi(G) = \varphi_r(G)$.

**Proof.** Let $t = \varphi_r(G)$. By Proposition 3.9, it suffices to prove that every minimal $b$-$t$-atom is feasible to have $\varphi(G) = \varphi_r(G)$. Let $A_t$ be a minimal $b$-$t$-atom. Our proof consists in extending the coloring of $A_t$ induced by $D_1, \ldots, D_t$ to $N(A_t)$ using colors from $\{1, \ldots, t\}$. Thus, we consider that the vertices of $A_t$ are already colored.

For a vertex $u \in N(A_t)$, we denote by $I_c(u)$ the set $\{i \in \{1, \ldots, t\} \mid \exists v \in N(u) \cap N[c_i]\}$. For a vertex $u \in V(A_t)$, we denote by $c^u$ a neighbor of $u$ in $C(A_t)$ if $u \notin C(A_t)$ or the vertex $u$ itself if $u \in C(A_t)$. Finally, we denote by $N[A_t]$ the set of vertices $V(A_t) \cup N(A_t)$. In the different cases, when we describe a cycle of length at most $k$ by $u_1 \ldots u_k$, it is assumed that, depending the configuration, consecutive symbols can denote the same vertex. In this proof, any considered vertex is supposed to be in $N[A_t]$. We begin by proving the following properties:

i) No vertex of $N(A_t)$ is adjacent to two vertices of $N[c_i]$, for $1 \leq i \leq t$;

ii) If $u, v \in N(A_t)$ and $i \in I_c(u) \cap I_c(v)$, then $u$ and $v$ are not adjacent and have no common neighbor in $N(A_t) - c_i$;

iii) If $u, v \in N[c_i]$ and $u', v' \in N[c_j]$, $u \neq v$, $u' \neq v'$, for some $i$ and $j$, $1 \leq i < j \leq t$, then the subgraph induced by $\{u, v, u', v'\}$ contains at most one edge.

i) If $u$ is adjacent to two vertices of $N[c_i]$, for some $i$, $1 \leq i \leq t$, then $u$ is in a cycle of length at most 4. This cycle contains $u, c_i$ and one or two vertices of $N[c_i]$.
ii) If \( u \) and \( v \) are adjacent or have a common neighbor, then \( u \) and \( v \) belong to a cycle of length at most 6. This cycle contains \( u, v, \) vertices of \( N[c_i] \) and possibly the common neighbor of \( u \) and \( v \) in \( N(A_t) - c_i, \) for \( i \) an integer such that \( i \in I_e(u) \cap I_e(v). \)

iii) If the subgraph induced by \( \{u, v, u', v'\} \) contains at least 2 edges, then there is a cycle of length at most 6 in \( G. \) This cycle is \( u-v-c_i \) if \( u \) and \( v \) are adjacent, \( u'-v'-c_j \) if \( u' \) and \( v' \) are adjacent or the cycle \( u-c_i-v-u'-v'-c_j, \) otherwise.

We are going to prove that either each vertex \( u \in N(A_t) \) can be colored with colors from \( \{1, \ldots, t\} \) or the graph \( G \) contains a \( b \cdot (t + 1) \)-atom (which contradicts \( \varphi_e(G) = t \). By properties i) and ii), any vertex of \( N(A_t) \) has at most \( t \) neighbors in \( N[A_t]. \) Hence we may suppose that any vertex \( u \in N(A_t) \) with less than \( t \) neighbors in \( N[A_t] \) is already colored and only consider vertices of \( N(A_t) \) with \( t \) neighbors in \( N[A_t]. \) For a vertex \( u \in N[A_t], \) a color \( i \) is said to be available for \( u \) if no vertex has color \( i \) in \( N(u) \cap N[A_t] \) (and therefore, \( u \) has no available color if the colors \( 1, \ldots, t \) are not available for \( u \)). Let \( N_u(A_t) \) be the set of vertices in \( N(A_t) \) with no available colors.

We define the following three sets:

- \( N_1 = \{u \in N(A_t) | N(u) \cap (V(A_t) \setminus C(A_t)) = \emptyset, N(u) \cap N(A_t) = \emptyset\}; \)

- \( N_2 = \{u \in N(A_t) | N(u) \cap (V(A_t) \setminus C(A_t)) = \emptyset, N(u) \cap N(A_t) \neq \emptyset\}; \)

- \( N_3 = \{u \in N(A_t) | N(u) \cap (V(A_t) \setminus C(A_t)) = \emptyset\}. \)

We can remark that \( N_1 \cup N_2 \cup N_3 = N(A_t). \)

In the remainder of the proof we will first consider the vertices of \( N_1; \) secondly the vertices of \( N_2; \) and finally the vertices of \( N_3. \)

**Case 1:** vertices of \( N_1. \)

Let \( u \) be a vertex of \( N_1. \) We recall that, by the above assumption, \( u \) has exactly \( t \) neighbors in \( A_t. \) Moreover, by Property i), \( |I_e(u)| = t. \) Let \( c_i \in C(A_t). \) We denote by \( A_i^u \) the vertices of \( N(c_i) \) which have a neighbor in \( N_u(A_t). \) Notice that a vertex \( v \in A_i^u \) can not have a neighbor \( x \) in \( V(A_t) \setminus \{c_i\} \) since otherwise it would create a cycle \( v-x-c_i-v'-u, \) for \( u \) the neighbor of \( v \) in \( N_1 \cap N_u(A_t) \) and \( v' \) the neighbor of \( u \) in \( N[c_i]. \) This cycle has length at most 5, contradicting \( g \geq 7. \) If for a vertex \( c_i \in C(A_t) \) we have \( |A_i^u| \geq 2, \) we exchange the colors of the vertices of \( A_i^u \) by doing a cyclic permutation of their colors. Afterwards, we obtain that some vertices of \( N_1 \cap N_u(A_t) \) have now an available color and we recolor them by any available color. Finally, we color the vertices of \( N_1, \) when possible, by any available color. Let \( N_{**}(A_t) \) be the set of the remaining uncolored vertices of \( N_1. \) In the following subcases, we recolor at most once the vertices of \( N[c_i], \) for \( i \in \{1, \ldots, t\}, \) since any two vertices of \( N_{**}(A_t) \) can not both have neighbors in \( N(c_i). \)
By considering that $N_{**}(A_t) \neq \emptyset$ (or else we have nothing more to do in Case 1), we can suppose that for every two integers $i, j, 1 \leq i \neq j \leq t$, we have $N[c_i] \cap N[c_j] = \emptyset$. Otherwise, if there exists a vertex $u \in N_{**}(A_t)$ and a vertex $w \in N[c_i] \cap N[c_j]$, there is a cycle $u-v-c_i-w-c_j-v'$ of length at most 6, for $v$ a neighbor of $u$ in $N[c_i]$ and $v'$ a neighbor of $u$ in $N[c_j]$.

Thus, we obtain that if $N_{**}(A_t) \neq \emptyset$, then every vertex $c_i \in C(A_t)$ has only one neighbor of color $j$, for $1 \leq i \neq j \leq t$, since otherwise it would contradict the minimality of $A_t$ (by removing one vertex of color $j$).

We then consider the two following subcases, for $u \in N_{**}(A_t)$.

**Subcase 1.1:** $u$ has exactly one neighbor in $V(A_t) \setminus C(A_t)$.
Let $v'$ be the neighbor of $u$ in $V(A_t) \setminus C(A_t)$ and let $c'$ be the color of $v'$. Notice that no vertex $x$ from $N[c']$ has a neighbor $y$ in $V(A_t) \setminus N[c']$, since otherwise it would create a cycle $u-v'\rightarrow x-y\rightarrow c'$ of length at most 6. Consequently, we can exchange the color of $v'$ with the color of one vertex from $N[c']$ and color $u$ by $c'$.

**Subcase 1.2:** $u$ has more than one neighbor in $V(A_t) \setminus C(A_t)$.
Let $v_1$ and $v_2$ be two neighbors of $u$ in $V(A_t) \setminus C(A_t)$. Let $c'$ be the color of $v_1$ and let $c''$ be the color of $v_2$.

If $v_1$ has a neighbor $x \in V(A_t) \setminus N[c''_1]$, then there exists a cycle $u-v_1-x-c''-v'$ in $G$, with $v'$ a neighbor of $c''$ in $N(u)$ (in the case $c''$ is not a neighbor of $u$). Similarly if $v_2$ has a neighbor in $V(A_t) \setminus N[c''_2]$, then there is a cycle of length at most 5 in $G$. Consequently, we can suppose that $v_1$ has no neighbor in $V(A_t) \setminus N[c''_1]$ and that $v_2$ has no neighbor in $V(A_t) \setminus N[c''_2]$. If there exists a vertex of $N[c''_1] \setminus \{v_1\}$ with no neighbor of color $c'$, then
we exchange the color of $v_1$ with the color of this vertex and color $u$ by $c'$. If there exists a vertex of $N(c^{v_2}) \setminus \{v_2\}$ with no neighbor of color $c''$, then we exchange the color of $v_2$ with the color of this vertex and color $u$ by $c''$. Thus, we may suppose that every vertex $w$ of $N(c^{v_1}) \setminus \{v_1\}$ (of $N(c^{v_2}) \setminus \{v_2\}$, respectively) has a neighbor $\overline{w}$ of color $c'$ ($c''$, respectively) in $V(A_t)$. We consider three subcases in order to color to $u$.

**Subcase 1.2.1:** the vertices $v_1$ and $c^{v_2}$ have the same color and the vertices $v_2$ and $c^{v_1}$ have the same color.

Notice that no vertex $w \in N(c^{v_1})$ is adjacent to $c^{v_2}$ since otherwise $u$-$v_1$-$c^{v_1}$-$w$-$c^{v_2}$-$v_2$ would be a cycle of length at most 6 in $G$. For the same reason, no vertex $w \in N(c^{v_2})$ is adjacent to $c^{v_1}$. Thus, by Property iii), no vertex $w \in N(c^{v_1}) \cup N(c^{v_2})$ has a neighbor $x \in V(A_t) \setminus (N(c^{v_1}) \cup N(c^{v_2}) \cup \overline{w})$, since there exists a vertex $y \in N(c^{w})$ with neighbor $\overline{y} \in N(c^{\overline{w}})$. There could exist two adjacent vertices $w$ and $w'$ with $w \in N(c^{v_1})$ and $w' \in N(c^{v_2})$. However, the vertex $w'$ has no neighbor of color $c''$ in $A_t$ since $w'$ and $v_2$ can not be adjacent and there does not exist a second vertex of color $c''$ in $N(c^{v_2})$. Consequently, we can exchange the color of $v_1$ with the color of $v_2$, the color of $c^{v_1}$ with the color of $c^{v_2}$ and afterward we can exchange the color of one vertex from $N(c^{v_1}) \setminus \{v_1\}$ with the color of $v_1$ and color $u$ by $c''$. The top of Figure 5 illustrates this recoloring process on a minimal b-4-atom fulfilling the hypothesis of Subcase 1.2.1.

**Subcase 1.2.2:** the vertices $v_1$ and $c^{v_2}$ do not have the same color.

Let $i$ be the color of $c^{v_1}$ and $j$ be the color of $c^{v_2}$. In this case, we exchange the color of $c^{v_2}$ with the color of $c^{v_1}$ and the color of the vertex $w$ of color $j$ in $N(c^{v_1})$ with the color of the vertex $w'$ of color $i$ in $N(c^{v_2})$. For this, we have to suppose that $w$ is not adjacent to a vertex of color $i$ and that $w'$ is not adjacent to a vertex of color $j$. For $t \geq 4$, such vertices $w$ and $w'$ exist since at most one vertex of $N(c^{v_1})$ has a neighbor of color $j$ (otherwise, it would contradict Property iii) since every vertex of $N(c^{v_1}) \setminus \{v_1\}$ has already a neighbor in $V(A_t)$ of color $c'$ and at most one vertex of $N(c^{v_2})$ has a neighbor of color $i$. If $t = 3$, then the only (up to isomorphism) b-3-atom with a coloring fulfilling all these hypothesis (up to color permutation) is illustrated at the bottom of Figure 5, along with the recoloring process. In this b-3-atom, no more edge can be added (otherwise, it would create a cycle of length at most 6).

**Subcase 1.2.3:** the vertices $v_2$ and $c^{v_1}$ do not have the same color.

We proceed as for the previous subcase by considering $v_2$ instead of $v_1$ and $c^{v_1}$ instead of $c^{v_2}$.

**Case 2:** vertices of $N_2$.

Since each pair of adjacent vertices $u, v \in N(A_t)$ satisfies Property ii), we obtain that $L_i(u) \cap L_c(v) = \emptyset$. We color each vertex $u \in N_2$ by a color $i \in L_i(u)$ such that $u$ and $c_i$ are not adjacent.
Case 3: vertices of $N_3$.

Notice that, by definition, a vertex of $C(A_t)$ has no available color. Let $u \in N_3$. We begin by coloring $u$ with any available color if it has some. If $u$ has no available color, there could exist a color $i$ such that every vertex of $N(u)$ with color $i$ has an available color (these vertices should be in $N(A_t)$). If such color $i$ exists, we recolor these vertices of color $i$ by any available color and give color $i$ to $u$. If such color $i$ does not exist, then the set of vertices at distance at most 2 from $u$ induces a $b-(t+1)$-atom with center $N[u]$. It can be noted that the recolored vertices are in $N(A_t)$ since $N(u) \cap V(A_t) \subseteq C(A_t)$.

We finish this proof by illustrating that the obtained coloring is a $b$-$t$-coloring of $N[A_t]$. In case 1, we have modified the coloring of $A_t$. However, since we have exchanged the colors of well-chosen vertices in order that every vertex of $C(A_t)$ still has neighbor of every color from $\{1, \ldots, t\}$ except its own color, this coloring remains a $b$-$t$-coloring. In case 3, we have only changed the color of vertices from $N(A_t)$.

\[ \square \]

We think that the previous theorem can be useful to determine the family of graphs of girth at least 7 satisfying $\varphi(G) = m(G)$. It has already been proven that graphs of girth at least 7 have $b$-chromatic number at least $m(G) - 1$ [7].

**Corollary 6.4.** Let $G$ be a graph of girth at least 7 and of order $n$ and let $t$ be an integer. There exists an algorithm in time $O(n^t)$ to determine if $\varphi(G) \geq t$.

## 7 Open questions

We conclude this article by listing few open questions.

1. For which family of graphs are the $b$-relaxed number and the $b$-chromatic number equal?

2. Does there exist an easy characterization of feasible $b$-$t$-atoms?

3. Does there exist an FPT algorithm, with parameter $t$, to determine if $\varphi(G) \geq t$?

**References**


