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A characterization of b-chromatic and partial Grundy numbers by induced subgraphs

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Abstract

Gyárfás et al. and Zaker have proven that the Grundy number of a graph \(G\) satisfies \(\Gamma(G) \geq t\) if and only if \(G\) contains an induced subgraph called a \(t\)-atom. The family of \(t\)-atoms has bounded order and contains a finite number of graphs. In this article, we introduce equivalents of \(t\)-atoms for b-coloring and partial Grundy coloring. This concept is used to prove that determining if \(\varphi(G) \geq t\) and \(\partial \Gamma(G) \geq t\) (under conditions for the b-coloring), for a graph \(G\), is in XP with parameter \(t\). We illustrate the utility of the concept of \(t\)-atoms by giving results on b-critical vertices and edges, on b-perfect graphs and on graphs of girth at least 7.

1 Introduction

Given a graph \(G\), a proper \(k\)-coloring of \(G\) is a surjective function \(c : V(G) \to \{1, \ldots, k\}\) such that \(c(u) \neq c(v)\) for any \(uv \in E(G)\); the color class \(V_i\) is the set \(\{u \in V | c(u) = i\}\) and a vertex \(v\) has color \(i\) if \(v \in V_i\). We denote by \(N(u)\) the set of neighbors of a vertex \(u\) and by \(N[u]\) the set \(N(u) \cup \{u\}\). A vertex \(v\) of color \(i\) is a Grundy vertex if it is adjacent to at least one vertex colored \(j\), for every \(j < i\). A Grundy \(k\)-coloring is a proper \(k\)-coloring such that every vertex is a Grundy vertex. The Grundy number of a graph \(G\), denoted by \(\Gamma(G)\), is the largest integer \(k\) such that there exists a Grundy \(k\)-coloring of \(G\) [10]. A partial Grundy \(k\)-coloring is a proper \(k\)-coloring such that every color class contains at least one Grundy vertex. The partial Grundy number of a graph \(G\), denoted by \(\partial \Gamma(G)\), is the largest integer \(k\) such that there exists a partial Grundy \(k\)-coloring of \(G\). Let \(G\) and \(G'\) be two graphs. By \(G \cup G'\) we denote the graph with vertex set \(V(G) \cup V(G')\) and edge set \(E(G) \cup E(G')\). Let \(m(G)\) be the largest integer

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Another coloring parameter with domination constraints on the colors is the b-chromatic number. In a proper-k-coloring, a vertex v of color i is a b-vertex if v is adjacent to at least one vertex colored j, 1 ≤ j ≠ i ≤ k. A b-k-coloring, also called b-coloring when k is not specified, is a proper k-coloring such that every color class contains at least one b-vertex. The b-chromatic number of a graph G, denoted by ϕ(G), is the largest integer k such that there exists a b-k-coloring of G. In this paper, we introduce the concept of b-relaxed number, denoted by ϕr(G). A b-k-relaxed coloring of G is a b-k-coloring of a subgraph of G. The b-relaxed number of G is ϕr(G) = maxH⊆G(ϕ(H)), for H an induced subgraph of G. Note that we have ϕ(G) ≤ ϕr(G) ≤ ∂Γ(G). The difference between ϕ(G) and ϕr(G) can be arbitrary large. Let Kn,n denotes the complete bipartite graph in which we remove n − 1 pairwise non incident edges (or n − 1 edges of a perfect matching in Kn,n) [1]. For this graph we have ϕ(Kn,n) = 2 and ϕr(Kn,n) = n as Figure 1 illustrates it (for n = 3).

The concept of b-coloring has been introduced by Irving and Manlove [16], and a large number of papers was published (see e.g. [8, 19]). The b-chromatic number of regular graphs has been investigated in a serie of papers ([6, 17, 20, 22]). Determining the b-chromatic number of a tight graph is NP-hard even for a connected bipartite graph [18] and a tight chordal graph [12].

In this paper, we study the decision problems b-COL, b-r-COL and pG-COL with parameter t from Table 1.

<table>
<thead>
<tr>
<th>Question</th>
<th>b-COL</th>
<th>b-r-COL</th>
<th>G-COL</th>
<th>pG-COL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complexity class</td>
<td>undetermined</td>
<td>XP</td>
<td>XP [23]</td>
<td>XP</td>
</tr>
</tbody>
</table>

Table 1: The different decision problems with input a graph G and parameter t and their complexity class.
function $f$ and a constant $c$. A decision problem is in XP with parameter $t$ if there exists an algorithm which resolves the problem in time $O(f(t) \cdot n^{g(t)})$, for an instance of size $n$ and two computable functions $f$ and $g$.

The concept of $t$-atom was introduced independently by Gyárfás et al. [11] and by Zaker [23]. The family of $t$-atoms is finite and the presence of a $t$-atom can be determined in polynomial time for a fixed $t$. The following definition is slightly different from the definitions of Gyárfás et al. or Zaker, insisting more on the construction of every $t$-atom (some $t$-atoms can not be obtained with the initial construction of Zaker).

**Definition 1.1** ([23]). The family of $t$-atoms is denoted by $\mathcal{A}_t^t$, for $t \geq 1$, and is defined by induction. The family $\mathcal{A}_1^t$ only contains $K_1$. A graph $G$ is in $\mathcal{A}_{t+1}^t$ if there exists a graph $G'$ in $\mathcal{A}_t^t$ and an integer $m$, $m \leq |V(G')|$, such that $G$ is composed of $G'$ and an independent set $I_m$ of order $m$, adding edges between $G'$ and $I_m$ such that every vertex in $G'$ is adjacent to at least one vertex in $I_m$.

Moreover, in the following sections, we say that a graph $G$ in a family of graphs $\mathcal{F}$ is minimal, if no graphs of $\mathcal{F}$ is a proper induced subgraph of $G$. For example, a minimal $t$-atom $A$ is a $t$-atom for which there are no $t$-atoms which are induced in $A$ other than itself.

**Theorem 1.1** ([11, 23]). A graph $G$ satisfies $\Gamma(G) \geq t$ if and only if it contains an induced minimal $t$-atom.

In this paper we prove equivalent theorems for b-relaxed number and partial Grundy number. In contrast with the minimal $t$-atoms, we can not define the minimal $t$-atoms for b-coloring as the smallest graphs such that $G$ satisfies $\varphi(G) = t$ (also called b-critical graphs).

The paper is organized as follows: Section 2 is devoted to the definition of $t$-atoms for the partial Grundy coloring. This concept allows us to prove that the partial Grundy coloring problem is in XP with parameter $t$. Section 3 is similar to Section 2 but for $b$-relaxed coloring. Section 4 is devoted to the concept of $b$-critical vertices and edges. Section 5 is about $b$-perfect graphs. Finally, Section 6 deals with graphs for which the $b$-relaxed and the $b$-chromatic numbers are equal.

## 2 Partial-Grundy-$t$-atoms: $t$-atoms for partial Grundy coloring

We start this section with the definition of $t$-atoms for partial Grundy coloring.

**Definition 2.1.** Given an integer $t$, a partial Grundy $t$-atom (or $pG$-$t$-atom, for short) is a graph $A$ whose vertex-set can be partitioned into $t$ sets $D_1, \ldots, D_t$, where $D_i$ contains a special vertex $c_i$ for each $i \in \{1, \ldots, t\}$ such that the following holds:

- For all $i \in \{1, \ldots, t\}$, $D_i$ is an independent set and $|D_i| \leq t - i + 1$;
Figure 2: The minimal pG-2-atom (on the left) and the three minimal pG-3-atoms (the numbers are the colors of the vertices and the surrounded vertices form the centers).

- For all $i \in \{2, \ldots, t\}$, $c_i$ has a neighbor in each of $D_1, \ldots, D_{i-1}$.

The set $\{c_1, \ldots, c_t\}$ is called the center of $A$ and denoted by $C(A)$.

Note that the sets $D_1, \ldots, D_t$ induce a partial Grundy coloring of the pG-$t$-atom. Figure 2 illustrates several pG-$t$-atoms (and their induced colorings) obtained using the previous definition.

**Observation 2.1.** For every pG-$t$-atom $G$, we have $|V(G)| \leq \frac{t(t+1)}{2}$.

**Lemma 2.2.** Let $t$ and $t'$ be two integers such that $1 \leq t' < t$. Every pG-$t$-atom contains a pG-$t'$-atom as induced subgraph.

*Proof.* Every pG-$t$-atom $G$ contains a pG-$t'$-atom $G'$: we can obtain $G'$ by removing every vertex in $D_k$, for $t' < k \leq t$, and by removing, afterwards, the vertices of $G'$ not adjacent to any vertex in $\{c_1, \ldots, c_{t'}\}$. □

Note that the only minimal pG-2-atom is $P_2$. The minimal pG-3-atoms are $C_3$, $P_4$ and $P_2 \cup P_3$. These graphs are illustrated in Figure 2.

**Theorem 2.3.** For a graph $G$, we have $\partial \Gamma(G) \geq t$ if and only if $G$ contains an induced minimal pG-$t$-atom.

*Proof.* Suppose that $\partial \Gamma(G) = t'$ with $t' \geq t$. By definition, there exists a partial Grundy coloring of $G$ with $t'$ colors. Let $u_1, \ldots, u_{t'}$ be a set of Grundy vertices, each in a different color class of $V(G)$. The graph induced by $N[u_1] \cup \ldots \cup N[u_{t'}]$ contains a pG-$t'$-atom. Hence, by Lemma 2.2, since $G$ contains an induced pG-$t'$-atom, then it also contains an induced minimal pG-$t$-atom.

Suppose $G$ contains an induced minimal pG-$t$-atom. Thus, the sets $D_1, \ldots, D_t$ induce a partial-Grundy coloring of this pG-$t$-atom. We can extend this coloring to a partial Grundy coloring of $G$ with at least $t$ colors in a greedy way by coloring the remaining vertices in any order, assigning to each of them the smallest color not used by its neighbors. □
Proposition 2.4. Let $G$ be a graph of order $n$ and let $t$ be an integer. There exists an algorithm in time $O(n^{\frac{t(t+1)}{2}})$ to determine if $\partial \Gamma(G) \geq t$. Hence, the problem $pG$-COL with parameter $t$ is in XP.

Proof. By Theorem 2.3, it suffices to verify that $G$ contains an induced minimal $pG$-$t$-atom to have $\partial \Gamma(G) \geq t$. Since the order of a minimal $pG$-$t$-atom is bounded by $\frac{(t+1)}{2}$, we obtain an algorithm in time $O(n^{\frac{t(t+1)}{2}})$.

We finish this section by determining every graph $G$ with $\partial \Gamma(G) = 2$.

Proposition 2.5. For a graph $G$ without isolated vertices, we have $\partial \Gamma(G) = 2$ if and only if $G = K_{n,m}$, for $n \geq 2$ and $m \geq 1$ or $G$ only contains isolated edges.

Proof. Zaker [23] has proven that $\Gamma(G) = 2$ if and only if $G$ is the disjoint union of copies of some $K_{n,m}$, for $n \geq 1$ and $m \geq 1$. Let $n$ and $m$ be positive integers. We can note that a graph containing a copy of $K_{n,m}$, for $n \geq 2$ and $m \geq 1$ and a copy of $K_{n,m}$, for $n \geq 1$ and $m \geq 1$ contains an induced $P_3 \cup P_2$, hence a $pG$-3-atom. Hence, if $\partial \Gamma(G) = 2$, then $G = K_{n,m}$, for $n \geq 2$ and $m \geq 1$ or $G$ only contains isolated edges.

Moreover, neither $K_{n,m}$ nor $P_2 \cup \ldots \cup P_2$ does contain an induced $C_3$, $P_4$ or $P_3 \cup P_2$. Hence, $\partial \Gamma(K_{n,m}) = 2$.

3 b-t-atoms: t-atoms for b-coloring

As in the previous section, we start this section with the definition of b-t-atoms (the notion of t-atom for b-coloring).

Definition 3.1. Given an integer $t$, a b-t-atom is a graph $A$ whose vertex-set can be partitioned into $t$ sets $D_1, \ldots, D_t$, where $D_i$ contains a special vertex $c_i$ for each $i \in \{1, \ldots, t\}$ such that the following holds:

- For each $i \in \{1, \ldots, t\}$, $D_i$ is an independent set and $|D_i| \leq t$;
- For all $i, j \in \{1, \ldots, t\}$, with $i \neq j$, $c_i$ has a neighbor in $D_j$.

The set $\{c_1, \ldots, c_t\}$ is called the center of $A$ and denoted by $C(A)$.

Note that the sets $D_1, \ldots, D_t$ induce a b-coloring of the b-t-atom. Figure 3 illustrates several b-t-atoms (and their induced coloring) obtained using the previous definition.

Observation 3.1. For every b-t-atom $G$, we have $|V(G)| \leq t^2$.

Lemma 3.2. Let $t$ and $t'$ be two integers such that $1 \leq t' < t$. Every b-t-atom contains a b-$t'$-atom as induced subgraph.

Proof. Every b-t-atom $G$ contains a b-$t'$-atom $G'$: we can obtain $G'$ by removing every vertex in $D_k$, for $t' < k \leq t$, and by removing, afterwards, the vertices not adjacent to any vertex in $\{c_1, \ldots, c_{t'}\}$.

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Figure 3: The minimal b-2-atom (on the left) and the five minimal b-3-atoms.

Note that the only minimal b-2-atom is $P_2$. The minimal b-3-atoms are $C_3$, $P_5$, $C_5$, $P_3 \cup P_4$ and $P_3 \cup P_3 \cup P_3$. These graphs are illustrated in Figure 3.

**Observation 3.3.** Every minimal pG-$t$-atom is an induced subgraph of a minimal b-$t$-atom or a minimal $t$-atom (an atom for the Grundy number).

**Proposition 3.4.** Let $G$ be a graph. If $\varphi(G) \geq t$, then $G$ contains an induced minimal b-$t$-atom.

**Proof.** Suppose that $\varphi(G) = t'$, with $t' \geq t$. Thus, there exists a b-coloring of $G$ with $t'$ colors. Let $u_1, \ldots, u_{t'}$ be a set of b-vertices, each in a different color class of $V(G)$. The graph induced by $N[u_1] \cup \ldots \cup N[u_{t'}]$ contains a b-$t'$-atom. Hence, by Lemma 3.2, since $G$ contains an induced b-$t'$-atom, then it also contains an induced minimal b-$t$-atom.

**Theorem 3.5.** For a graph $G$, we have $\varphi_r(G) \geq t$ if and only if $G$ contains an induced minimal b-$t$-atom.

**Proof.** Suppose that the graph $G$ contains an induced b-$t$-atom $A$. Since $A$ admits, by definition, a b-$t$-coloring, we have $\varphi_r(G) \geq t$. Using Proposition 3.4, we obtain the converse.

**Definition 3.2.** Let $G$ be a graph. For an induced subgraph $A$ of $G$, let $N(A) = \{v \in V(G) \setminus V(A) \mid uv \in E(G), u \in V(A)\}$. A b-$t$-atom $A$ is feasible in $G$ if there exists a b-$t$-coloring of $V(A)$ that can be extended to the vertices of $N(A)$ without using new colors.

**Proposition 3.6.** Let $G$ be a graph. If $G$ contains an induced feasible minimal b-$t$-atom and no induced feasible minimal b-$t'$-atom, for $t' > t$, then $\varphi_r(G) = t$.

**Proof.** Suppose that $G$ contains an induced feasible minimal b-$t$-atom $A$ and no b-$t$-coloring of $G$ exists. We begin by considering that the vertices of $A \cup N(A)$ are already colored with $t$ colors. We can note that, by assumption, no coloring of $A \cup N(A)$ (from the definition) can be extended to the whole graph using only $t$ colors. Let $t'$ be the largest integer such that the coloring can not be extended
Proof. By Theorem 3.5, it suffices to verify that problem $b$-$r$-$\text{COL}$ with parameter $t$ exists an algorithm in time $O(n^t)$ to determine if a graph has $b$-relaxed number at least $\varphi_r(G)$ if $G$ is complete even if $t$ is fixed. Let $G$ be a graph. If $\varphi(G) = t$, then $G$ contains an induced feasible minimal $b$-$t$-atom and no induced feasible minimal $b$-$t'$-atom, for $t' > t$.

Proof. Suppose $\varphi(G) = t$. By Proposition 3.4, $G$ contains an induced minimal $b$-$t$-atom. If no induced minimal $b$-$t$-atom is feasible, then there exists no $b$-$t$-coloring of $G$, a contradiction.

A direct consequence of Proposition 3.6 and Proposition 3.7 is the following.

Theorem 3.8. For a graph $G$, we have $\varphi(G) = t$ if and only if $G$ contains an induced feasible minimal $b$-$t$-atom and no induced feasible minimal $b$-$t'$-atom, for $t' > t$.

The following proposition will be useful in the last section.

Proposition 3.9. Let $G$ be a graph and let $t = \varphi_r(G)$. If every minimal $b$-$t$-atom is feasible in $G$, then $\varphi(G) = \varphi_r(G)$.

Proof. Since $t = \varphi_r(G)$, $G$ does not contain a $b$-$(t+1)$-atom. Thus, by Proposition 3.6, we obtain $\varphi(G) = t$.

Note that the problem of determining if a graph has a $b$-$t$-coloring is NP-complete even if $t$ is fixed [21]. However, it does not imply that determining if $\varphi(G) = t$ for a graph $G$ is NP-complete. In contrast with the $b$-chromatic number, determining if a graph has $b$-relaxed number at least $t$ is in XP.

Proposition 3.10. Let $G$ be a graph of order $n$ and let $t$ be an integer. There exists an algorithm in time $O(n^t^2)$ to determine if $\varphi_r(G) \geq t$. In particular, the problem $b$-$r$-$\text{COL}$ with parameter $t$ is in XP.

Proof. By Theorem 3.5, it suffices to verify that $G$ contains an induced minimal $b$-$t$-atom to determine if $\varphi_r(G) \geq t$. By Observation 3.1, the order of a minimal $b$-$t$-atom is bounded by $t^2$. Thus, we obtain an algorithm in time $O(n^t^2)$. Another NP-complete problem is to determine the $b$-spectrum of a graph $G$ [2], i.e. the set of integers $k$ such that $G$ is $b$-$k$-colorable. For a graph $G$ satisfying $\varphi(G) = \varphi_r(G)$, our algorithm can be used. Thus, proving that for a class of graphs, every graph $G$ satisfies $\varphi(G) = \varphi_r(G)$, implies that the problem $b$-$\text{COL}$ with parameter $t$ is in XP for this class of graphs.
4 b-critical vertices and edges

The concept of b-critical vertices and b-critical edges has been introduced recently and since five years a large number of articles are considering this subject [1, 4, 5, 9, 24]. In this section, we illustrate how this notion is strongly connected with the concept of b-t-atom.

Definition 4.1 ([4, 9]). Let \( G \) be a graph. A vertex \( v \) of \( G \) is b-critical if \( \varphi(G - v) < \varphi(G) \). An edge \( e \) is b-critical if \( \varphi(G - e) < \varphi(G) \). A vertex \( v \) (edge \( e \), respectively) in a graph \( G \) is a b-trap, if there exists a b-\( t \)-atom of \( G \) that becomes feasible by removing \( v \) (\( e \), respectively).

Proposition 4.1. Let \( G \) be a graph. A vertex \( v \) is b-critical if and only if it is in every feasible minimal b-\( \varphi(G) \)-atom and \( v \) is not a b-\( \varphi(G) \)-trap.

Proof. Let \( t = \varphi(G) \). First, if \( v \) is not in a feasible minimal b-\( t \)-atom, then \( \varphi(G - v) = t \) and \( v \) is not b-critical. If \( v \) is a b-\( t \)-trap, then, by definition, \( \varphi(G - v) = t \). Second, suppose \( v \) is not a b-\( t \)-trap. If \( v \) is in every feasible minimal b-\( t \)-atom, then, since every minimal b-\( t \)-atom in \( G \) does not contain any other feasible minimal b-\( t \)-atom as induced subgraph, \( G - v \) does not contain a feasible minimal b-\( t \)-atom. Thus, \( v \) is b-critical.

Corollary 4.2. If a graph \( G \) contains two induced feasible minimal b-\( \varphi(G) \)-atoms with disjoint set of vertices, then it contains no b-critical vertex.

Proposition 4.3. Let \( G \) be a graph and \( v \) be a vertex of \( V(G) \). If \( \varphi(G - v) > \varphi(G) \), then \( G \) contains a minimal b-\( \varphi(G - v) \)-atom which is not feasible. If \( \varphi(G - v) < \varphi(G) - 1 \), then \( G - v \) contains no feasible minimal b-\( t \)-atom, for \( \varphi(G - v) < t \leq \varphi(G) \).

Proof. Note that every b-\( t \)-atom contained in \( G - v \) is also contained in \( G \), for any integer \( t \). Thus, if \( \varphi(G - v) > \varphi(G) \), then \( G \) contains a b-\( \varphi(G - v) \)-trap and consequently a minimal b-\( \varphi(G - v) \)-atom which is not feasible. Moreover, if \( \varphi(G - v) < \varphi(G) - 1 \) and \( G - v \) contains a feasible b-\( t \)-atom for \( \varphi(G - v) < t \leq \varphi(G) \), then \( \varphi(G - v) \geq t \).

In [1], Balakrishnan and Raj have proved the following theorem.

Theorem 4.4 ([1]). Let \( G \) be a graph and \( v \) be a vertex of \( V(G) \). We have \( \varphi(G) - \lfloor \frac{|V(G)|}{2} \rfloor + 2 \leq \varphi(G - v) \leq \varphi(G) + \lfloor \frac{|V(G)|}{2} \rfloor - 2 \).

Moreover, they have determined the families of graphs for which there exists a vertex \( v \) such that \( \varphi(G - v) = \varphi(G) - \lfloor \frac{|V(G)|}{2} \rfloor + 2 \) or \( \varphi(G - v) = \varphi(G) + \lfloor \frac{|V(G)|}{2} \rfloor - 2 \). In contrast with the b-chromatic number, we have the following property about the b-relaxed number.

Proposition 4.5. Let \( G \) be a graph. If a vertex \( v \) is b-critical, then \( \varphi_r(G - v) = \varphi_r(G) - 1 \).

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Proof. By Proposition 4.1, \( v \) is in every \( b-\varphi(G) \)-atom. Let \( i \) be the integer associated to \( v \) in the construction of this \( b-\varphi(G) \)-atom. By removing the vertices with associated integer \( i \), we obtain a \( b-(\varphi(G) - 1) \)-atom and thus \( \varphi_r(G - v) = \varphi_r(G) - 1 \).

Note that this proposition was already proved for trees [4].

**Lemma 4.6.** Let \( G \) be a graph with \( 4 \leq |V(G)| \leq 5 \) and \( E(G) \neq \emptyset \). We have \( \varphi_r(G - v) = \varphi_r(G) + \left\lfloor \frac{|V(G)|}{2} \right\rfloor - 2 \), for every vertex \( v \) of \( V(G) \), if and only if \( G \) contains two disjoint edges but no induced minimal \( b-3 \)-atom.

**Proof.** We can note that we have \( \varphi_r(G - v) = \varphi_r(G) + \left\lfloor \frac{|V(G)|}{2} \right\rfloor - 2 \) if and only if \( \varphi_r(G - v) = \varphi_r(G) \).

First, if \( G \) contains no minimal \( b-3 \)-atom and contains an edge, then \( \varphi_r(G) = 2 \). Moreover, if \( G \) contains two disjoint edges, then for any vertex \( v \), \( G - v \) contains \( P_2 \) and \( \varphi_r(G - v) = 2 \).

Second, suppose that for every vertex \( v \), \( \varphi_r(G - v) = \varphi_r(G) \). The only minimal \( b-3 \)-atoms that contains at most five vertices are \( K_3 \), \( C_5 \) and \( P_5 \). Moreover, the only minimal \( b-4 \)-atoms and \( b-5 \)-atoms that contain at most five vertices are \( K_4 \) and \( K_5 \). We are going to show that \( G \) is not one of these graphs.

**Case 1:** \( \varphi_r(G) = 5 \). If \( G \) is a \( K_5 \), then, by removing any vertex \( v \), we obtain \( \varphi_r(G - v) = 4 \).

**Case 2:** \( \varphi_r(G) = 4 \). If \( G \) is a \( K_4 \), then, by removing any vertex \( v \), we obtain \( \varphi_r(G - v) = 3 \). If \( G \) contains an induced \( K_4 \), \( |V(G)| = 5 \) and \( G \) is not \( K_5 \), then there exists a vertex \( v \) such \( G - v \) has no induced \( K_4 \) and \( \varphi_r(G - v) = 3 \).

**Case 3:** \( \varphi_r(G) = 3 \). If \( G \) contains an induced \( K_3 \) and no induced \( K_4 \), then, since the induced \( K_3 \) in \( G \) have a common vertex \( v \), we obtain \( \varphi_r(G - v) = 2 \). Moreover, if \( G \) is \( P_3 \) or \( C_5 \), then, by removing any vertex \( v \), we obtain \( \varphi_r(G - v) = 2 \).

Thus, we can suppose that \( \varphi_r(G) = 2 \). If \( G \) contains only edges with a common vertex \( v \), then \( \varphi_r(G - v) = 1 \). Hence, \( G \) contains no \( b-3 \)-atom and contain two disjoint edges.

The following theorem is a generalization of a conjecture of Blidia et al. [3] for the parameter \( \varphi_r \). Note that the graphs \( P_3 \), \( C_4 \) and \( P_2 \cup P_2 \) do not contain any induced minimal \( b-3 \)-atom and contain two disjoint edges.

**Theorem 4.7.** Let \( G \) be a graph. We have \( \varphi_r(G - v) = \varphi_r(G) + \left\lfloor \frac{|V(G)|}{2} \right\rfloor - 2 \), for every vertex \( v \) of \( V(G) \), if and only one of these conditions is true about \( G \):

i) \( G \) is \( P_2 \) or \( C_3 \).

ii) \( E(G) = \emptyset \) and \( 4 \leq |V(G)| \leq 5 \).

iii) \( 4 \leq |V(G)| \leq 5 \) and \( G \) contains two disjoint edges but no \( b-3 \)-atom.
Proof. Note that if $|V(G)| \geq 6$, then, by Proposition 4.5, we can not have $\varphi_r(G - v) = \varphi_r(G) + \lfloor |V(G)|/2 \rfloor - 2$. Note also that if $G$ contains only one vertex, then it can not satisfy $\varphi_r(G - v) = \varphi_r(G) + \lfloor |V(G)|/2 \rfloor - 2$.

First, if $2 \leq |V(G)| \leq 3$, then we have $\varphi_r(G - v) = \varphi_r(G) - 1$ if and only if $G$ contains no edges, then $\varphi_r(G) = 1$ and for any vertex $v$, $\varphi_r(G - v) = 1$. The third condition is obtained by Lemma 4.6.

Definition 4.2. Let $t$ be a positive integer and $A$ be a $b$-$t$-atom. An edge $e$ is $b$-atom-critical in $A$ if $A - e$ is not a $b$-$t$-atom.

Proposition 4.8. Let $G$ be a graph. An edge $e$ is $b$-critical if and only if it is $b$-atom-critical in every feasible minimal $b$-$\varphi(G)$-atom and $e$ is not a $b$-$\varphi(G)$-trap.

Proof. Let $t = \varphi(G)$. First, if $e$ is not $b$-atom-critical in a feasible minimal $b$-$t$-atom, then $G - e$ contains a feasible minimal $b$-$t$-atom and $\varphi(G - e) = t$. If $e$ is a $b$-$t$-trap, then, by definition, $\varphi(G - e) = t$. Second, suppose that $e$ is not a $b$-$t$-trap. If $e$ is $b$-atom-critical in every feasible minimal $b$-$t$-atom, then, since every feasible minimal $b$-$t$-atom in $G$ does not contain any other feasible minimal $b$-$t$-atom as subgraph in $G - e$, the graph $G - e$ does not contain a feasible minimal $b$-$t$-atom. Thus, $e$ is $b$-critical.

Corollary 4.9. If a graph $G$ contains two induced feasible minimal $b$-$\varphi(G)$-atoms with disjoint sets of $b$-atom-critical edges, then $G$ contains no $b$-critical edge.

5 b-perfect graphs

A b-perfect graph is a graph for which every induced subgraph satisfies that its $b$-chromatic number is equal to its chromatic number. More generally, we present the following definitions.

Definition 5.1 ([13]). A graph $G$ is $b$-$\chi$-$k$-bounded, for $k$ a positive integer, if $\varphi(G') - \chi(G') \leq k$, for every induced subgraph $G'$ of $G$. A graph $G$ is a $\chi$-$k$-unbounded $b$-atom, for $k$ a positive integer, if $\varphi(G) - \chi(G) > k$ and $G$ is a $b$-$t$-atom for some integer $t$. A graph $G$ is an imperfect $b$-atom, for $k$ a positive integer, if $\varphi(G) > \chi(G)$ and $G$ is a $b$-$t$-atom for some integer $t$.

Hoang et al. [14] characterized b-perfect graphs by giving the family $F$ of forbidden induced subgraphs depicted in Figure 4. We recall the following theorem:

Theorem 5.1 ([14]). A graph is b-perfect if and only if it contains no graph from $F$ as induced subgraph.

Note that every graph in the family $F$ is a $b$-$t$-atom for some $t$. More precisely, $F_1$, $F_2$ and $F_3$ are the only minimal bipartite $b$-3-atoms. The remaining
graphs are minimal b-4-atoms that do not contain $F_1$, $F_2$ and $F_3$ as induced subgraph and which admit a proper coloring with three colors (as mentioned in [15]). We can state the following property about b-\(\phi\)-atoms.

**Theorem 5.2.** Let \(k\) be a positive integer. A graph \(G\) is not \(b-\chi-k\)-bounded if and only if it contains a minimal \(\chi-k\)-unbounded b-atom.

**Proof.** First, if \(G\) contains a minimal \(\chi-k\)-unbounded b-atom, then, by definition, \(G\) is not \(\chi-k\)-bounded.

Second, suppose \(G\) is not \(b-\chi-k\)-bounded. Then, there exists an induced subgraph \(A\) of \(G\) of minimal order which is not \(b-\chi-k\)-bounded. By removing vertices of \(A\) we can only decrease the chromatic number. Thus, by removing vertices we can obtain a b-\(\phi(A)\)-atom which is \(\chi-k\)-unbounded.

**Corollary 5.3.** The graphs with \(b\)-chromatic number \(t\) which are \(b-\chi-k\)-bounded, for fixed integers \(k\) and \(t\), can be defined by forbidding a finite family of induced subgraphs: the \(\chi-k\)-unbounded b-atoms. Hence, a graph \(G\) is \(b\)-perfect if and only if it does not contain imperfect b-atoms.
Let \( b\chi \)-BOUNDED be the following decision problem and let \( k \) be an integer, with \( 0 \leq k < \varphi(G) \).

**b-\( \chi \)-k-BOUNDED**

**Instance**: A graph \( G \).

**Question**: Does \( \varphi(G) - \chi(G) \geq k ? \)

By Corollary 5.3, we obtain the following corollary:

**Corollary 5.4.** Let \( G \) be a graph and \( k \) be an integer, with \( 0 \leq k < \varphi(G) \).

There exists an algorithm in time \( O(n^{\varphi(G)^2}) \) to solve \( b\chi\)-k-BOUNDED.

Since a graph \( G \) is b-perfect if and only if it does not contain imperfect b-atoms, we have the following theorem:

**Theorem 5.5.** The number of imperfect b-atoms is finite. A graph is an imperfect b-atom if and only if it is in the family \( F \) (Figure 4).

The previous theorem is a consequence of Theorem 5.1. Remark that if we can prove that every minimal b-4-atom except \( K_4 \) contains an induced subgraph of the family \( F \), then, using Theorem 5.2, we obtain another proof of Theorem 5.1.

### 6 b-chromatic and b-relaxed chromatic numbers

In this section we consider the b-relaxed number relatively to the b-chromatic number and prove equality for trees and graphs of girth at least 7.

**Lemma 6.1.** A minimal b-t-atom has at most \( t \) connected components.

**Proof.** Suppose that a minimal b-t-atom \( G \) has more than \( t \) connected components. By definition, at least one connected component \( A \) of \( G \) does not contain a vertex of \( C(G) \). Since \( G - A \) is also a b-atom, \( G \) is not minimal. \( \square \)

Note that a minimal b-t-atom \( G \) contains a center \( C(G) \) and the remaining vertices of \( G \) are neighbors of vertices of \( C(G) \).

**Proposition 6.2.** For a tree \( T \), we have \( \varphi(T) = \varphi_r(T) \).

**Proof.** Let \( t = \varphi_r(T) \). By Proposition 3.9, it suffices to prove that every minimal b-t-atom is feasible to have \( \varphi(T) = \varphi_r(T) \). Let \( T' \) be a minimal b-t-atom and let \( N[T'] = V(T') \cup N(T') \). By Lemma 6.1, \( T' \) has at most \( t \) connected components. Let \( u \) be a vertex of \( N(T') \) with a maximal number of neighbors in \( N[T'] \). Since \( T' \) has at most \( t \) connected components and \( T \) is a tree, \( u \) has at most \( t \) neighbors in \( N[T'] \).

Our proof consists in extending the coloring of \( T' \) induced by \( D_1, \ldots, D_t \) to \( N(T') \) using colors from \( \{1, \ldots, t\} \). For \( t = 2 \), the proof is trivial since the only minimal b-2-atom is \( P_2 \) and we can easily extend the coloring to \( N(P_2) \). Thus we can suppose that \( t \geq 3 \). If \( u \) has at most \( t - 1 \) neighbors in \( N[T'] \),
The vertices of $N$ connected components of $T$ vertex is supposed to be in $N$ consecutive symbols can denote the same vertex. In this proof, any considered length at most $k$.

Finally, the vertices of $N(u) \cap N(T)$ can be recolored if we have obtained an improper coloring by recoloring $w$. 

The girth of a graph $G$ is the length of a smallest cycle in $G$. We finish this paper by proving that when a graph $G$ has sufficiently large girth, we have $\varphi(G) = \varphi_r(G)$, thus extending Proposition 6.2.

**Theorem 6.3.** Let $G$ be a graph with girth $g$ and $\varphi_r(G) \geq 3$. If $g \geq 7$, then $\varphi(G) = \varphi_r(G)$.

**Proof.** Let $t = \varphi_r(G)$. By Proposition 3.9, it suffices to prove that every minimal $b$-$t$-atom is feasible to have $\varphi(G) = \varphi_r(G)$. Let $A_t$ be a minimal $b$-$t$-atom. Our proof consists in extending the coloring of $A_t$ induced by $D_1, \ldots, D_t$ to $N(A_t)$ using colors from $\{1, \ldots, t\}$. Thus, we consider that the vertices of $A_t$ are already colored.

For a vertex $u \in N(A_t)$, we denote by $I_c(u)$ the set $\{i \in \{1, \ldots, t\} \mid \exists v \in N(u) \cap N[c_i]\}$. For a vertex $u \in V(A_t)$, we denote by $e_a$ a neighbor of $u$ in $C(A_t)$ if $u \notin C(A_t)$ or the vertex $u$ itself if $u \in C(A_t)$. Finally, we denote by $N[A_t]$, the set of vertices $V(A_t) \cup N(A_t)$. In the different cases, when we describe a cycle of length at most $k$ by $u_1 \ldots u_k$, it is assumed that, depending the configuration, consecutive symbols can denote the same vertex. In this proof, any considered vertex is supposed to be in $N[A_t]$. We begin by proving the following properties:

1. No vertex of $N(A_t)$ is adjacent to two vertices of $N[c_i]$, for $1 \leq i \leq t$;
2. If $u, v \in N(A_t)$ and $i \in I_c(u) \cap I_c(v)$, then $u$ and $v$ are not adjacent and have no common neighbor in $N(A_t) - c_i$;
3. If $u, v \in N[c_i]$ and $u', v' \in N[c_j]$, $u \neq v$, $u' \neq v'$, for some $i$ and $j$, $1 \leq i < j \leq t$, then the subgraph induced by $\{u, v, u', v'\}$ contains at most one edge;
4. If $u$ is adjacent to two vertices of $N[c_i]$, for some $i$, $1 \leq i \leq t$, then $u$ is in a cycle of length at most 4. This cycle contains $u, c_i$ and one or two vertices of $N[c_i]$.

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ii) If \( u \) and \( v \) are adjacent or have a common neighbor, then \( u \) and \( v \) belong to a cycle of length at most 6. This cycle contains \( u, v \), vertices of \( N[c_i] \) and possibly the common neighbor of \( u \) and \( v \) in \( N(A_t) - c_i \), for \( i \) an integer such that \( i \in I(c) \cap I(v) \).

iii) If the subgraph induced by \( \{u, v, u', v'\} \) contains at least 2 edges, then there is a cycle of length at most 6 in \( G \). This cycle is \( u-v-c_i \) if \( u \) and \( v \) are adjacent, \( u'-v'-c_j \) if \( u' \) and \( v' \) are adjacent or the cycle \( u-c_i-v-u'-v'-c_j \), otherwise.

We are going to prove that either each vertex \( u \in N(A_t) \) can be colored with colors from \( \{1, \ldots, t\} \) or the graph \( G \) contains a \( b-(t+1) \)-atom (which contradicts \( \varphi_t(G) = t \)). By properties i) and ii), any vertex of \( N(A_t) \) has at most \( t \) neighbors in \( N[A_t] \). Hence we may suppose that any vertex \( u \in N(A_t) \) with less than \( t \) neighbors in \( N[A_t] \) is already colored and only consider vertices of \( N(A_t) \) with \( t \) neighbors in \( N[A_t] \). For a vertex \( u \in N[A_t] \), a color \( i \) is said to be available for \( u \) if no vertex has color \( i \) in \( N(u) \cap N[A_t] \) (and therefore, \( u \) has no available color if the colors 1, \ldots, \( t \) are not available for \( u \)). Let \( N_u(A_t) \) be the set of vertices in \( N(A_t) \) with no available colors.

We define the following three sets:

- \( N_1 = \{u \in N(A_t) | N(u) \cap (V(A_t) \setminus C(A_t)) \neq \emptyset, N(u) \cap N(A_t) = \emptyset\} \);
- \( N_2 = \{u \in N(A_t) | N(u) \cap (V(A_t) \setminus C(A_t)) = \emptyset, N(u) \cap N(A_t) \neq \emptyset\} \);
- \( N_3 = \{u \in N(A_t) | N(u) \cap (V(A_t) \setminus C(A_t)) = \emptyset\} \).

We can remark that \( N_1 \cup N_2 \cup N_3 = N(A_t) \).

In the remainder of the proof we will first consider the vertices of \( N_1 \); secondly the vertices of \( N_2 \); and finally the vertices of \( N_3 \).

**Case 1:** vertices of \( N_1 \).

Let \( u \) be a vertex of \( N_1 \). We recall that, by the above assumption, \( u \) has exactly \( t \) neighbors in \( A_t \). Moreover, by Property i), \( |I_c(u)| = t \). Let \( c_i \in C(A_t) \). We denote \( A^*_i \) the vertices of \( N(c_i) \) which have a neighbor in \( N_u(A_t) \). Notice that a vertex \( v \in A^*_i \) can not have a neighbor \( x \) in \( V(A_t) \setminus \{c_i\} \) since otherwise it would create a cycle \( v-x-c_i-v' \), for \( u \) the neighbor of \( v \) in \( N_1 \cap N_u(A_t) \) and \( v' \) the neighbor of \( u \) in \( N[c_i] \). This cycle has length at most 5, contradicting \( g \geq 7 \). If for a vertex \( c_i \in C(A_t) \) we have \( |A^*_i| \geq 2 \), we exchange the colors of the vertices of \( A^*_i \) by doing a cyclic permutation of their colors. Afterwards, we obtain that some vertices of \( N_1 \cap N_u(A_t) \) have now an available color and we recolor them by any available color. Finally, we color the vertices of \( N_1 \), when possible, by any available color. Let \( N_u(A_t) \) be the set of the remaining uncolored vertices of \( N_1 \). In the following subcases, we recolor at most once the vertices of \( N[c_i] \), for \( i \in \{1, \ldots, t\} \), since any two vertices of \( N_u(A_t) \) can not both have neighbors in \( N(c_i) \).
Subcase 1.1: Let \( v' \) be the neighbor of \( u \) in \( V(A_t) \setminus C(A_t) \) and let \( c' \) be the color of \( v' \). Notice that no vertex \( x \) from \( N[c'^*] \) has a neighbor \( y \) in \( V(A_t) \setminus N[c'^*] \), since otherwise it would create a cycle \( u-v'x-c'^*x-y-c^y \) of length at most 6. Consequently, we can exchange the color of \( v' \) with the color of one vertex from \( N(c'^*) \) and color \( u \) by \( c' \).

Subcase 1.2: \( u \) has more than one neighbor in \( V(A_t) \setminus C(A_t) \).
Let \( v_1 \) and \( v_2 \) be two neighbors of \( u \) in \( V(A_t) \setminus C(A_t) \). Let \( c' \) be the color of \( v_1 \) and let \( c'' \) be the color of \( v_2 \).
If \( v_1 \) has a neighbor \( x \) in \( V(A_t) \setminus N[c'^*] \), then there exists a cycle \( u-v_1x-c'^*x-v' \) in \( G \), with \( v' \) a neighbor of \( c'^* \) in \( N(u) \) (in the case \( c'^* \) is not a neighbor of \( u \)). Similarly if \( v_2 \) has a neighbor in \( V(A_t) \setminus N[c'^*] \), then there is a cycle of length at most 5 in \( G \). Consequently, we can suppose that \( v_1 \) has no neighbor in \( V(A_t) \setminus N[c'^*] \) and that \( v_2 \) has no neighbor in \( V(A_t) \setminus N[c'^*] \).
If there exists a vertex of \( N(c'^*) \setminus \{v_1\} \) with no neighbor of color \( c' \), then
we exchange the color of \( v_1 \) with the color of this vertex and color \( u \) by \( c' \). If there exists a vertex of \( N(c^{v_2}) \setminus \{v_2\} \) with no neighbor of color \( c'' \), then we exchange the color of \( v_2 \) with the color of this vertex and color \( u \) by \( c'' \). Thus, we may suppose that every vertex \( w \) of \( N(c^{v_1}) \setminus \{v_1\} \) (of \( N(c^{v_2}) \setminus \{v_2\} \), respectively) has a neighbor \( w \) of color \( c' \) (\( c'' \), respectively) in \( V(A_t) \). We consider three subcases in order to color to \( u \).

**Subcase 1.2.1:** the vertices \( v_1 \) and \( c^{v_2} \) have the same color and the vertices \( v_2 \) and \( c^{v_1} \) have the same color.

Notice that no vertex \( w \in N(c^{v_1}) \) is adjacent to \( c^{v_2} \) since otherwise \( u \)-\( v_1 \)-\( c^{v_1} \)-\( w \)-\( c^{v_2} \)-\( v_2 \) would be a cycle of length at most 6 in \( G \). For the same reason, no vertex \( w \in N(c^{v_2}) \) is adjacent to \( c^{v_1} \). Thus, by Property iii), no vertex \( w \in N(c^{v_1}) \cup N(c^{v_2}) \) has a neighbor \( x \in V(A_t) \setminus (N[c^{v_1}] \cup N[c^{v_2}] \cup \{w\}) \), since there exists a vertex \( y \in N(c^{w}) \) with neighbor \( w \in N(c^{v_2}) \).

There could exist two adjacent vertices \( w \) and \( w' \) with \( w \in N(c^{v_1}) \) and \( w' \in N(c^{v_2}) \). However, the vertex \( w' \) has no neighbor of color \( c'' \) in \( A_t \) since \( w' \) and \( v_2 \) can not be adjacent and there does not exist a second vertex of color \( c'' \) in \( N(c^{v_2}) \). Consequently, we can exchange the color of \( v_1 \) with the color of \( v_2 \), the color of \( c^{v_1} \) with the color of \( c^{v_2} \) and afterward we can exchange the color of one vertex from \( N(c^{v_1}) \setminus \{v_1\} \) with the color of \( v_1 \) and color \( u \) by \( c'' \). The top of Figure 5 illustrates this recoloring process on a minimal b-4-atom fulfilling the hypothesis of Subcase 1.2.1.

**Subcase 1.2.2:** the vertices \( v_1 \) and \( c^{v_2} \) do not have the same color.

Let \( i \) be the color of \( c^{v_1} \) and \( j \) be the color of \( c^{v_2} \). In this case, we exchange the color of \( c^{v_2} \) with the color of \( c^{v_1} \) and the color of the vertex \( w \) of color \( j \) in \( N(c^{v_1}) \) with the color of the vertex \( w' \) of color \( i \) in \( N(c^{v_2}) \). For this, we have to suppose that \( w \) is not adjacent to a vertex of color \( i \) and that \( w' \) is not adjacent to a vertex of color \( j \). For \( t \geq 4 \), such vertices \( w, w' \) exist since at most one vertex of \( N(c^{v_1}) \) has a neighbor of color \( j \) (otherwise, it would contradict Property iii) since every vertex of \( N(c^{v_1}) \setminus \{v_1\} \) has already a neighbor in \( V(A_t) \) of color \( c' \) and at most one vertex of \( N(c^{v_2}) \) has a neighbor of color \( i \). If \( t = 3 \), then the only (up to isomorphism) b-3-atom with a coloring fulfilling all these hypothesis (up to color permutation) is illustrated at the bottom of Figure 5, along with the recoloring process. In this b-3-atom, no more edge can be added (otherwise, it would create a cycle of length at most 6).

**Subcase 1.2.3:** the vertices \( v_2 \) and \( c^{v_1} \) do not have the same color.

We proceed as for the previous subcase by considering \( v_2 \) instead of \( v_1 \) and \( c^{v_1} \) instead of \( c^{v_2} \).

**Case 2:** vertices of \( N_2 \).

Since each pair of adjacent vertices \( u, v \in N(A_t) \) satisfies Property ii), we obtain that \( I_c(u) \cap I_c(v) = \emptyset \). We color each vertex \( u \in N_2 \) by a color \( i \in I_c(u) \) such that \( u \) and \( c_i \) are not adjacent.
Case 3: vertices of $N_3$.
Notice that, by definition, a vertex of $C(A_t)$ has no available color. Let $u \in N_3$. We begin by coloring $u$ with any available color if it has some. If $u$ has no available color, there could exist a color $i$ such that every vertex of $N(u)$ with color $i$ has an available color (these vertices should be in $N(A_t)$). If such color $i$ exists, we recolor these vertices of color $i$ by any available color and give color $i$ to $u$. If such color $i$ does not exist, then the set of vertices at distance at most 2 from $u$ induces a $b-(t+1)$-atom with center $N[u]$. It can be noted that the recolored vertices are in $N(A_t)$ since $N(u) \cap V(A_t) \subseteq C(A_t)$.

We finish this proof by illustrating that the obtained coloring is a $b$-$t$-coloring of $N[A_t]$. In case 1, we have modified the coloring of $A_t$. However, since we have exchanged the colors of well-chosen vertices in order that every vertex of $C(A_t)$ still has neighbor of every color from $\{1, \ldots, t\}$ except its own color, this coloring remains a $b$-$t$-coloring. In case 3, we have only changed the color of vertices from $N(A_t)$.

We think that the previous theorem can be useful to determine the family of graphs of girth at least 7 satisfying $\varphi(G) = m(G)$. It has already been proven that graphs of girth at least 7 have b-chromatic number at least $m(G) - 1$ [7].

\textbf{Corollary 6.4.} Let $G$ be a graph of girth at least 7 and of order $n$ and let $t$ be an integer. There exists an algorithm in time $O(n t^2)$ to determine if $\varphi(G) \geq t$.

\section{Open questions}

We conclude this article by listing few open questions.

1. For which family of graphs are the $b$-relaxed number and the $b$-chromatic number equal?

2. Does there exist an easy characterization of feasible $b$-$t$-atoms?

3. Does there exist an FPT algorithm, with parameter $t$, to determine if $\varphi(G) \geq t$?

\section{References}


