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HAL Id: hal-01157902
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Submitted on 28 Apr 2016

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A characterization of b-chromatic and partial Grundy numbers by induced subgraphs

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April 28, 2016

Abstract

Gyárfás et al. and Zaker have proven that the Grundy number of a graph $G$ satisfies $\Gamma(G) \geq t$ if and only if $G$ contains an induced subgraph called a $t$-atom. The family of $t$-atoms has bounded order and contains a finite number of graphs. In this article, we introduce equivalents of $t$-atoms for b-coloring and partial Grundy coloring. This concept is used to prove that determining if $\varphi(G) \geq t$ and $\partial\Gamma(G) \geq t$ (under conditions for the b-coloring), for a graph $G$, is in XP with parameter $t$. We illustrate the utility of the concept of $t$-atoms by giving results on b-critical vertices and edges, on b-perfect graphs and on graphs of girth at least 7.

1 Introduction

Given a graph $G$, a proper $k$-coloring of $G$ is a surjective function $c : V(G) \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for any $uv \in E(G)$; the color class $V_i$ is the set $\{u \in V|c(u) = i\}$ and a vertex $v$ has color $i$ if $v \in V_i$. We denote by $N(u)$ the set of neighbors of a vertex $u$ and by $N[u]$ the set $N(u) \cup \{u\}$. A vertex $v$ of color $i$ is a Grundy vertex if it is adjacent to at least one vertex colored $j$, for every $j < i$. A Grundy $k$-coloring is a proper $k$-coloring such that every vertex is a Grundy vertex. The Grundy number of a graph $G$, denoted by $\Gamma(G)$, is the largest integer $k$ such that there exists a Grundy $k$-coloring of $G$ \cite{10}. A partial Grundy $k$-coloring is a proper $k$-coloring such that every color class contains at least one Grundy vertex. The partial Grundy number of a graph $G$, denoted by $\partial\Gamma(G)$, is the largest integer $k$ such that there exists a partial Grundy $k$-coloring of $G$. Let $G$ and $G'$ be two graphs. By $G \cup G'$ we denote the graph with vertex set $V(G) \cup V(G')$ and edge set $E(G) \cup E(G')$. Let $m(G)$ be the largest integer
Figure 1: The graph $K_{3,3}$ with $\varphi(K_{3,3}) = 2$ (on the left) and $\varphi_r(K_{3,3}) = 3$ (on the right).

A decision problem is in FPT with parameter $t$ if there exists an algorithm which resolves the problem in time $O(f(t) \cdot n^c)$, for an instance of size $n$, a computable
function \( f \) and a constant \( c \). A decision problem is in XP with parameter \( t \) if there exists an algorithm which resolves the problem in time \( O(f(t) \ n^{g(t)}) \), for an instance of size \( n \) and two computable functions \( f \) and \( g \).

The concept of \( t \)-atom was introduced independently by Gyárfás et al. [11] and by Zaker [23]. The family of \( t \)-atoms is finite and the presence of a \( t \)-atom can be determined in polynomial time for a fixed \( t \). The following definition is slightly different from the definitions of Gyárfás et al. or Zaker, insisting more on the construction of every \( t \)-atom (some \( t \)-atoms cannot be obtained with the initial construction of Zaker).

**Definition 1.1** ([23]). The family of \( t \)-atoms is denoted by \( \mathcal{A}_t \), for \( t \geq 1 \), and is defined by induction. The family \( \mathcal{A}_1 \) only contains \( K_1 \). A graph \( G \) is in \( \mathcal{A}_{t+1} \) if there exists a graph \( G' \) in \( \mathcal{A}_t \) and an integer \( m \), \( m \leq |V(G')| \), such that \( G \) is composed of \( G' \) and an independent set \( I_m \) of order \( m \), adding edges between \( G' \) and \( I_m \) such that every vertex in \( G' \) is adjacent to at least one vertex in \( I_m \).

Moreover, in the following sections, we say that a graph \( G \) in a family of graphs \( \mathcal{F} \) is minimal, if no graphs of \( \mathcal{F} \) is a proper induced subgraph of \( G \). For example, a minimal \( t \)-atom \( A \) is a \( t \)-atom for which there are no \( t \)-atoms which are induced in \( A \) other than itself.

**Theorem 1.1** ([11, 23]). A graph \( G \) satisfies \( \Gamma(G) \geq t \) if and only if it contains an induced minimal \( t \)-atom.

In this paper we prove equivalent theorems for \( b \)-relaxed number and partial Grundy number. In contrast with the minimal \( t \)-atoms, we cannot define the minimal \( t \)-atoms for \( b \)-coloring as the smallest graphs such that \( G \) satisfies \( \varphi(G) = t \) (also called \( b \)-critical graphs).

The paper is organized as follows: Section 2 is devoted to the definition of \( t \)-atoms for the partial Grundy coloring. This concept allows us to prove that the partial Grundy coloring problem is in XP with parameter \( t \). Section 3 is similar to Section 2 but for \( b \)-relaxed-coloring. Section 4 is devoted to the concept of \( b \)-critical vertices and edges. Section 5 is about \( b \)-perfect graphs. Finally, Section 6 deals with graphs for which the \( b \)-relaxed and the \( b \)-chromatic numbers are equal.

## 2 Partial-Grundy-\( t \)-atoms: \( t \)-atoms for partial Grundy coloring

We start this section with the definition of \( t \)-atoms for partial Grundy coloring.

**Definition 2.1.** Given an integer \( t \), a partial Grundy \( t \)-atom (or \( pG \)-\( t \)-atom, for short) is a graph \( A \) whose vertex-set can be partitioned into \( t \) sets \( D_1, \ldots, D_t \), where \( D_i \) contains a special vertex \( c_i \) for each \( i \in \{1, \ldots, t\} \) such that the following holds:

- For all \( i \in \{1, \ldots, t\} \), \( D_i \) is an independent set and \( |D_i| \leq t - i + 1 \);
• For all $i \in \{2, \ldots, t\}$, $c_i$ has a neighbor in each of $D_1, \ldots, D_{i-1}$.

The set $\{c_1, \ldots, c_t\}$ is called the center of $A$ and denoted by $C(A)$.

Note that the sets $D_1, \ldots, D_t$ induce a partial Grundy coloring of the $pG$-$t$-atom. Figure 2 illustrates several $pG$-$t$-atoms (and their induced colorings) obtained using the previous definition.

Observation 2.1. For every $pG$-$t$-atom $G$, we have $|V(G)| \leq \frac{t(t+1)}{2}$.

Lemma 2.2. Let $t$ and $t'$ be two integers such that $1 \leq t' < t$. Every $pG$-$t$-atom contains a $pG$-$t'$-atom as induced subgraph.

Proof. Every $pG$-$t$-atom $G$ contains a $pG$-$t'$-atom $G'$: we can obtain $G'$ by removing every vertex in $D_k$, for $t' < k \leq t$, and by removing, afterwards, the vertices of $G'$ not adjacent to any vertex in $\{c_1, \ldots, c_{t'}\}$.

Note that the only minimal $pG$-$2$-atom is $P_2$. The minimal $pG$-$3$-atoms are $C_3$, $P_4$ and $P_2 \cup P_3$. These graphs are illustrated in Figure 2.

Theorem 2.3. For a graph $G$, we have $\partial \Gamma(G) \geq t$ if and only if $G$ contains an induced minimal $pG$-$t$-atom.

Proof. Suppose that $\partial \Gamma(G) = t'$ with $t' \geq t$. By definition, there exists a partial Grundy coloring of $G$ with $t'$ colors. Let $u_1, \ldots, u_{t'}$ be a set of Grundy vertices, each in a different color class of $V(G)$. The graph induced by $N[u_1] \cup \ldots \cup N[u_{t'}]$ contains a $pG$-$t'$-atom. Hence, by Lemma 2.2, since $G$ contains an induced $pG$-$t'$-atom, then it also contains an induced minimal $pG$-$t$-atom.

Suppose $G$ contains an induced minimal $pG$-$t$-atom. Thus, the sets $D_1, \ldots, D_t$ induce a partial-Grundy coloring of this $pG$-$t$-atom. We can extend this coloring to a partial Grundy coloring of $G$ with at least $t$ colors in a greedy way by coloring the remaining vertices in any order, assigning to each of them the smallest color not used by its neighbors.
Proposition 2.4. Let \( G \) be a graph of order \( n \) and let \( t \) be an integer. There exists an algorithm in time \( O(n^{\frac{t(t+1)}{2}}) \) to determine if \( \partial \Gamma(G) \geq t \). Hence, the problem \( pG\text{-COL} \) with parameter \( t \) is in XP.

Proof. By Theorem 2.3, it suffices to verify that \( G \) contains an induced minimal \( pG\text{-}t \)-atom to have \( \partial \Gamma(G) \geq t \). Since the order of a minimal \( pG\text{-}t \)-atom is bounded by \( \frac{(t+1)^2}{2} \), we obtain an algorithm in time \( O(n^{\frac{t(t+1)}{2}}) \).

We finish this section by determining every graph \( G \) with \( \partial \Gamma(G) = 2 \).

Proposition 2.5. For a graph \( G \) without isolated vertices, we have \( \partial \Gamma(G) = 2 \) if and only if \( G = K_{n,m} \), for \( n \geq 2 \) and \( m \geq 1 \) or \( G \) only contains isolated edges.

Proof. Zaker [23] has proven that \( \Gamma(G) = 2 \) if and only if \( G \) is the disjoint union of copies of some \( K_{n,m} \), for \( n \geq 1 \) and \( m \geq 1 \). Let \( n \) and \( m \) be positive integers. We can note that a graph containing a copy of \( K_{n,m} \), for \( n \geq 2 \) and \( m \geq 1 \) and a copy of \( K_{n,m} \), for \( n \geq 1 \) and \( m \geq 1 \) contains an induced \( P_3 \cup P_2 \), hence a \( pG\text{-}3 \)-atom. Hence, if \( \partial \Gamma(G) = 2 \), then \( G = K_{n,m} \), for \( n \geq 2 \) and \( m \geq 1 \) or \( G \) only contains isolated edges.

Moreover, neither \( K_{n,m} \) nor \( P_2 \cup \ldots \cup P_2 \) does contain an induced \( C_3 \), \( P_4 \) or \( P_3 \cup P_2 \). Hence, \( \partial \Gamma(K_{n,m}) = 2 \). \( \square \)

3 b-t-atoms: t-atoms for b-coloring

As in the previous section, we start this section with the definition of b-t-atoms (the notion of t-atom for b-coloring).

Definition 3.1. Given an integer \( t \), a b-t-atom is a graph \( A \) whose vertex-set can be partitioned into \( t \) sets \( D_1, \ldots, D_t \), where \( D_i \) contains a special vertex \( c_i \) for each \( i \in \{1, \ldots, t\} \) such that the following holds:

- For each \( i \in \{1, \ldots, t\} \), \( D_i \) is an independent set and \( |D_i| \leq t \);
- For all \( i, j \in \{1, \ldots, t\} \), with \( i \neq j \), \( c_i \) has a neighbor in \( D_j \).

The set \( \{c_1, \ldots, c_t\} \) is called the center of \( A \) and denoted by \( C(A) \).

Note that the sets \( D_1, \ldots, D_t \) induce a b-coloring of the b-t-atom. Figure 3 illustrates several b-t-atoms (and their induced coloring) obtained using the previous definition.

Observation 3.1. For every b-t-atom \( G \), we have \( |V(G)| \leq t^2 \).

Lemma 3.2. Let \( t \) and \( t' \) be two integers such that \( 1 \leq t' < t \). Every b-t-atom contains a b-t'-atom as induced subgraph.

Proof. Every b-t-atom \( G \) contains a b-t'-atom \( G' \): we can obtain \( G' \) by removing every vertex in \( D_k \), for \( t' < k \leq t \), and by removing, afterwards, the vertices not adjacent to any vertex in \( \{c_1, \ldots, c_{t'}\} \). \( \square \)
Note that the only minimal b-2-atom is $P_2$. The minimal b-3-atoms are $C_3$, $P_5$, $C_5$, $P_3 \cup P_4$ and $P_3 \cup P_3 \cup P_3$. These graphs are illustrated in Figure 3.

**Observation 3.3.** Every minimal $pG$-$t$-atom is an induced subgraph of a minimal $b$-$t$-atom or a minimal $t$-atom (an atom for the Grundy number).

**Proposition 3.4.** Let $G$ be a graph. If $\varphi(G) \geq t$, then $G$ contains an induced minimal b-$t$-atom.

**Proof.** Suppose that $\varphi(G) = t'$, with $t' \geq t$. Thus, there exists a b-coloring of $G$ with $t'$ colors. Let $u_1, \ldots, u_{t'}$ be a set of b-vertices, each in a different color class of $V(G)$. The graph induced by $N[u_1] \cup \ldots \cup N[u_{t'}]$ contains a b-$t'$-atom. Hence, by Lemma 3.2, since $G$ contains an induced b-$t'$-atom, then it also contains an induced minimal b-$t$-atom.

**Theorem 3.5.** For a graph $G$, we have $\varphi_r(G) \geq t$ if and only if $G$ contains an induced minimal b-$t$-atom.

**Proof.** Suppose that the graph $G$ contains an induced b-$t$-atom $A$. Since $A$ admits, by definition, a b-$t$-coloring, we have $\varphi_r(G) \geq t$. Using Proposition 3.4, we obtain the converse.

**Definition 3.2.** Let $G$ be a graph. For an induced subgraph $A$ of $G$, let $N(A) = \{v \in V(G) \setminus V(A) \mid uv \in E(G) \text{, } u \in V(A)\}$. A b-$t$-atom $A$ is feasible in $G$ if there exists a b-$t$-coloring of $V(A)$ that can be extended to the vertices of $N(A)$ without using new colors.

**Proposition 3.6.** Let $G$ be a graph. If $G$ contains an induced feasible minimal b-$t$-atom and no induced feasible minimal b-$t'$-atom, for $t' > t$, then $\varphi(G) = t$.

**Proof.** Suppose that $G$ contains an induced feasible minimal b-$t$-atom $A$ and no b-$t$-coloring of $G$ exists. We begin by considering that the vertices of $A \cup N(A)$ are already colored with $t$ colors. We can note that, by assumption, no coloring of $A \cup N(A)$ (from the definition) can be extended to the whole graph using only $t$ colors. Let $t'$ be the largest integer such that the coloring can not be extended.
By Theorem 3.5, it suffices to verify that 

**Proof.** Suppose \( \varphi(G) = t \). By Proposition 3.4, \( G \) contains an induced minimal \( b \)-\( t \)-atom. If no induced minimal \( b \)-\( t \)-atom is feasible, then there exists no \( b \)-\( t \)-coloring of \( G \), a contradiction. 

A direct consequence of Proposition 3.6 and Proposition 3.7 is the following.

**Theorem 3.8.** For a graph \( G \), we have \( \varphi(G) = t \) if and only if \( G \) contains an induced feasible minimal \( b \)-\( t \)-atom and no induced feasible minimal \( b \)-\( t' \)-atom, for \( t' > t \).

The following proposition will be useful in the last section.

**Proposition 3.9.** Let \( G \) be a graph and let \( t = \varphi_r(G) \). If every minimal \( b \)-\( t \)-atom is feasible in \( G \), then \( \varphi(G) = \varphi_r(G) \).

**Proof.** Since \( t = \varphi_r(G) \), \( G \) does not contain a \( b \)-\( (t + 1) \)-atom. Thus, by Proposition 3.6, we obtain \( \varphi(G) = t \). 

Note that the problem of determining if a graph has a \( b \)-\( t \)-coloring is NP-complete even if \( t \) is fixed [21]. However, it does not imply that determining if \( \varphi(G) \geq t \) for a graph \( G \) is NP-complete. In contrast with the \( b \)-chromatic number, determining if a graph has \( b \)-relaxed number at least \( t \) is in XP.

**Proposition 3.10.** Let \( G \) be a graph of order \( n \) and let \( t \) be an integer. There exists an algorithm in time \( O(n^{t^2}) \) to determine if \( \varphi_r(G) \geq t \). In particular, the problem \( b \)-\( r \)-\( \text{COL} \) with parameter \( t \) is in XP.

**Proof.** By Theorem 3.5, it suffices to verify that \( G \) contains an induced minimal \( b \)-\( t \)-atom to determine if \( \varphi_r(G) \geq t \). By Observation 3.1, the order of a minimal \( b \)-\( t \)-atom is bounded by \( t^2 \). Thus, we obtain an algorithm in time \( O(n^{t^2}) \). 

Another NP-complete problem is to determine the \( b \)-spectrum of a graph \( G \) [2], i.e., the set of integers \( k \) such that \( G \) is \( b \)-\( k \)-colorable. For a graph \( G \) satisfying \( \varphi(G) = \varphi_r(G) \), our algorithm can be used. Thus, proving that for a class of graphs, every graph \( G \) satisfies \( \varphi(G) = \varphi_r(G) \), implies that the problem \( b \)-\( \text{COL} \) with parameter \( t \) is in XP for this class of graphs.
4 b-critical vertices and edges

The concept of b-critical vertices and b-critical edges has been introduced recently and since five years a large number of articles are considering this subject [1, 4, 5, 9, 24]. In this section, we illustrate how this notion is strongly connected with the concept of b-t-atom.

Definition 4.1 ([4, 9]). Let $G$ be a graph. A vertex $v$ of $G$ is b-critical if $\varphi(G - v) < \varphi(G)$. An edge $e$ is b-critical if $\varphi(G - e) < \varphi(G)$. A vertex $v$ (edge $e$, respectively) in a graph $G$ is a b-t-trap, if there exists a b-t-atom of $G$ that becomes feasible by removing $v$ (e, respectively).

Proposition 4.1. Let $G$ be a graph. A vertex $v$ is b-critical if and only if it is in every feasible minimal b-\(\varphi(G)\)-atom and $v$ is not a b-\(\varphi(G)\)-trap.

Proof. Let $t = \varphi(G)$. First, if $v$ is not in a feasible minimal b-t-atom, then $\varphi(G - v) = t$ and $v$ is not b-critical. If $v$ is a b-t-trap, then, by definition, $\varphi(G - v) = t$. Second, suppose $v$ is not a b-t-trap. If $v$ is in every feasible minimal b-t-atom, then, since every minimal b-t-atom in $G$ does not contain any other feasible minimal b-t-atom as induced subgraph, $G - v$ does not contain a feasible minimal b-t-atom. Thus, $v$ is b-critical.

Corollary 4.2. If a graph $G$ contains two induced feasible minimal b-\(\varphi(G)\)-atoms with disjoint set of vertices, then it contains no b-critical vertex.

Proposition 4.3. Let $G$ be a graph and $v$ be a vertex of $V(G)$. If $\varphi(G - v) > \varphi(G)$, then $G$ contains a minimal b-\(\varphi(G - v)\)-atom which is not feasible. If $\varphi(G - v) < \varphi(G) - 1$, then $G - v$ contains no feasible minimal b-t-atom, for $\varphi(G - v) < t \leq \varphi(G)$.

Proof. Note that every b-t-atom contained in $G - v$ is also contained in $G$, for any integer $t$. Thus, if $\varphi(G - v) > \varphi(G)$, then $G$ contains a b-\(\varphi(G - v)\)-trap and consequently a minimal b-\(\varphi(G - v)\)-atom which is not feasible. Moreover, if $\varphi(G - v) < \varphi(G) - 1$ and $G - v$ contains a feasible b-t-atom for $\varphi(G - v) < t \leq \varphi(G)$, then $\varphi(G - v) \geq t$.

In [1], Balakrishnan and Raj have proved the following theorem.

Theorem 4.4 ([1]). Let $G$ be a graph and $v$ be a vertex of $V(G)$. We have $\varphi(G) - \lfloor \frac{|V(G)|}{2} \rfloor + 2 \leq \varphi(G - v) \leq \varphi(G) + \lfloor \frac{|V(G)|}{2} \rfloor - 2$.

Moreover, they have determined the families of graphs for which there exists a vertex $v$ such that $\varphi(G - v) = \varphi(G) - \lfloor \frac{|V(G)|}{2} \rfloor + 2$ or $\varphi(G - v) = \varphi(G) + \lfloor \frac{|V(G)|}{2} \rfloor - 2$. In contrast with the b-chromatic number, we have the following property about the b-relaxed number.

Proposition 4.5. Let $G$ be a graph. If a vertex $v$ is b-critical, then $\varphi_r(G - v) = \varphi_r(G) - 1$. 
Proof. By Proposition 4.1, $v$ is in every $b$-$\varphi(G)$-atom. Let $i$ be the integer associated to $v$ in the construction of this $b$-$\varphi(G)$-atom. By removing the vertices with associated integer $i$, we obtain a $b$-$(\varphi(G) - 1)$-atom and thus $\varphi_r(G - v) = \varphi_r(G) - 1$. \hfill \Box

Note that this proposition was already proved for trees [4].

Lemma 4.6. Let $G$ be a graph with $4 \leq |V(G)| \leq 5$ and $E(G) \neq \emptyset$. We have $\varphi_r(G - v) = \varphi_r(G) + \floor{\frac{|V(G)|}{2}} - 2$, for every vertex $v$ of $V(G)$, if and only if $G$ contains two disjoint edges but no induced minimal $b$-$3$-atom.

Proof. We can note that we have $\varphi_r(G - v) = \varphi_r(G) + \floor{\frac{|V(G)|}{2}} - 2$ if and only if $\varphi_r(G - v) = \varphi_r(G)$.

First, if $G$ contains no minimal $b$-$3$-atom and contains an edge, then $\varphi_r(G) = 2$. Moreover, if $G$ contains two disjoint edges, then for any vertex $v$, $G - v$ contains $P_2$ and $\varphi_r(G - v) = 2$.

Second, suppose that for every vertex $v$, $\varphi_r(G - v) = \varphi_r(G)$. The only minimal $b$-$3$-atoms that contains at most five vertices are $K_3$, $C_5$ and $P_5$. Moreover, the only minimal $b$-$4$-atoms and $b$-$5$-atoms that contain at most five vertices are $K_3$ and $K_5$. We are going to show that $G$ is not one of these graphs

Case 1: $\varphi_r(G) = 5$. If $G$ is a $K_5$, then, by removing any vertex $v$, we obtain $\varphi_r(G - v) = 4$.

Case 2: $\varphi_r(G) = 4$. If $G$ is a $K_4$, then, by removing any vertex $v$, we obtain $\varphi_r(G - v) = 3$. If $G$ contains an induced $K_4$, $|V(G)| = 5$ and $G$ is not $K_5$, then there exists a vertex $v$ such $G - v$ has no induced $K_4$ and $\varphi_r(G - v) = 3$.

Case 3: $\varphi_r(G) = 3$. If $G$ contains an induced $K_3$ and no induced $K_4$, then, since the induced $K_3$ in $G$ have a common vertex $v$, we obtain $\varphi_r(G - v) = 2$. Moreover, if $G$ is $P_3$ or $C_5$, then, by removing any vertex $v$, we obtain $\varphi_r(G - v) = 2$.

Thus, we can suppose that $\varphi_r(G) = 2$. If $G$ contains only edges with a common vertex $v$, then $\varphi_r(G - v) = 1$. Hence, $G$ contains no $b$-$3$-atom and contain two disjoint edges. \hfill \Box

The following theorem is a generalization of a conjecture of Blidia et al. [3] for the parameter $\varphi_r$. Note that the graphs $P_4$, $C_4$ and $P_2 \cup P_2$ do not contain any induced minimal $b$-$3$-atom and contain two disjoint edges.

Theorem 4.7. Let $G$ be a graph. We have $\varphi_r(G - v) = \varphi_r(G) + \floor{\frac{|V(G)|}{2}} - 2$, for every vertex $v$ of $V(G)$, if and only one of these conditions is true about $G$:

i) $G$ is $P_2$ or $C_3$.

ii) $E(G) = \emptyset$ and $4 \leq |V(G)| \leq 5$.

iii) $4 \leq |V(G)| \leq 5$ and $G$ contains two disjoint edges but no $b$-$3$-atom.
Proof. Note that if $|V(G)| \geq 6$, then, by Proposition 4.5, we can not have $\varphi_r(G - v) = \varphi_r(G) + \lceil |V(G)|/2 \rceil - 2$. Note also that if $G$ contains only one vertex, then it can not satisfy $\varphi_r(G - v) = \varphi_r(G) + \lceil |V(G)|/2 \rceil - 2$.

First, if $2 \leq |V(G)| \leq 3$, then we have $\varphi_r(G - v) = \varphi_r(G) - 1$ if and only if $G$ is a minimal $b$-$t$-atom. Hence, if and only if $G$ is $P_2$ or $C_3$. Second, if $G$ contains no edges, then $\varphi_r(G) = 1$ and for any vertex $v$, $\varphi_r(G - v) = 1$. The third condition is obtained by Lemma 4.6.

**Definition 4.2.** Let $t$ be a positive integer and $A$ be a $b$-$t$-atom. An edge $e$ is $b$-atom-critical in $A$ if $A - e$ is not a $b$-$t$-atom.

**Proposition 4.8.** Let $G$ be a graph. An edge $e$ is $b$-critical if and only if it is $b$-atom-critical in every feasible minimal $b$-$\varphi(G)$-atom and $e$ is not a $b$-$\varphi(G)$-trap.

**Proof.** Let $t = \varphi(G)$. First, if $e$ is not $b$-atom-critical in a feasible minimal $b$-$t$-atom, then $G - e$ contains a feasible minimal $b$-$t$-atom and $\varphi(G - e) = t$. If $e$ is a $b$-$t$-trap, then, by definition, $\varphi(G - e) = t$. Second, suppose that $e$ is not a $b$-$t$-trap. If $e$ is $b$-atom-critical in every feasible minimal $b$-$t$-atom, then, since every feasible minimal $b$-$t$-atom in $G$ does not contain any other feasible minimal $b$-$t$-atom as subgraph in $G - e$, the graph $G - e$ does not contain a feasible minimal $b$-$t$-atom. Thus, $e$ is $b$-critical.

**Corollary 4.9.** If a graph $G$ contains two induced feasible minimal $b$-$\varphi(G)$-atoms with disjoint sets of $b$-atom-critical edges, then $G$ contains no $b$-critical edge.

## 5 b-perfect graphs

A b-perfect graph is a graph for which every induced subgraph satisfies that its $b$-chromatic number is equal to its chromatic number. More generally, we present the following definitions.

**Definition 5.1 ([13]).** A graph $G$ is $b$-$\chi$-$k$-bounded, for $k$ a positive integer, if $\varphi(G') - \chi(G') \leq k$, for every induced subgraph $G'$ of $G$. A graph $G$ is a $\chi$-$k$-unbounded $b$-atom, for $k$ a positive integer, if $\varphi(G) - \chi(G) > k$ and $G$ is a $b$-$t$-atom for some integer $t$. A graph $G$ is an imperfect $b$-atom, for $k$ a positive integer, if $\varphi(G) > \chi(G)$ and $G$ is a $b$-$t$-atom for some integer $t$.

Hoang et al. [14] characterized b-perfect graphs by giving the family $\mathcal{F}$ of forbidden induced subgraphs depicted in Figure 4. We recall the following theorem:

**Theorem 5.1 ([14]).** A graph is b-perfect if and only if it contains no graph from $\mathcal{F}$ as induced subgraph.

Note that every graph in the family $\mathcal{F}$ is a $b$-$t$-atom for some $t$. More precisely, $F_1$, $F_2$ and $F_3$ are the only minimal bipartite $b$-3-atoms. The remaining
graphs are minimal b-4-atoms that do not contain $F_1$, $F_2$ and $F_3$ as induced subgraph and which admit a proper coloring with three colors (as mentioned in [15]). We can state the following property about b-t-atoms.

**Theorem 5.2.** Let $k$ be a positive integer. A graph $G$ is not b-$\chi$-$k$-bounded if and only if it contains a minimal $\chi$-$k$-unbounded b-atom.

**Proof.** First, if $G$ contains a minimal $\chi$-$k$-unbounded b-atom, then, by definition, $G$ is not $\chi$-$k$-bounded.

Second, suppose $G$ is not b-$\chi$-$k$-bounded. Then, there exists an induced subgraph $A$ of $G$ of minimal order which is not b-$\chi$-$k$-bounded. By removing vertices of $A$ we can only decrease the chromatic number. Thus, by removing vertices we can obtain a b-$\varphi(A)$-atom which is $\chi$-$k$-unbounded.

**Corollary 5.3.** The graphs with b-chromatic number $t$ which are b-$\chi$-$k$-bounded, for fixed integers $k$ and $t$, can be defined by forbidding a finite family of induced subgraphs: the $\chi$-$k$-unbounded b-atoms. Hence, a graph $G$ is b-perfect if and only if it does not contain imperfect b-atoms.
Let $b$-χ-BOUNDED be the following decision problem and let $k$ be an integer, with $0 \leq k < \varphi(G)$.

**b-χ-k-BOUNDED**

**Instance**: A graph $G$.

**Question**: Does $\varphi(G) - \chi(G) \geq k$?

By Corollary 5.3, we obtain the following corollary:

**Corollary 5.4.** Let $G$ be a graph and $k$ be an integer, with $0 \leq k < \varphi(G)$. There exists an algorithm in time $O(n^{\varphi(G)^2})$ to solve $b$-χ-k-BOUNDED.

Since a graph $G$ is b-perfect if and only if it does not contain imperfect b-atoms, we have the following theorem:

**Theorem 5.5.** The number of imperfect b-atoms is finite. A graph is an imperfect b-atom if and only if it is in the family $F$ (Figure 4).

The previous theorem is a consequence of Theorem 5.1. Remark that if we can prove that every minimal b-4-atom except $K_4$ contains an induced subgraph of the family $F$, then, using Theorem 5.2, we obtain another proof of Theorem 5.1.

## 6 b-chromatic and b-relaxed chromatic numbers

In this section we consider the b-relaxed number relatively to the b-chromatic number and prove equality for trees and graphs of girth at least 7.

**Lemma 6.1.** A minimal b-t-atom has at most $t$ connected components.

**Proof.** Suppose that a minimal b-t-atom $G$ has more than $t$ connected components. By definition, at least one connected component $A$ of $G$ does not contain a vertex of $C(G)$. Since $G - A$ is also a b-t-atom, $G$ is not minimal.

Note that a minimal b-t-atom $G$ contains a center $C(G)$ and the remaining vertices of $G$ are neighbors of vertices of $C(G)$.

**Proposition 6.2.** For a tree $T$, we have $\varphi(T) = \varphi_r(T)$.

**Proof.** Let $t = \varphi_r(T)$. By Proposition 3.9, it suffices to prove that every minimal b-t-atom is feasible to have $\varphi(T) = \varphi_r(T)$. Let $T'$ be a minimal b-t-atom and let $N[T'] = V(T') \cup N(T')$. By Lemma 6.1, $T'$ has at most $t$ connected components. Let $u$ be a vertex of $N(T')$ with a maximal number of neighbors in $N[T']$. Since $T'$ has at most $t$ connected components and $T$ is a tree, $u$ has at most $t$ neighbors in $N[T']$.

Our proof consists in extending the coloring of $T'$ induced by $D_1, \ldots, D_t$ to $N(T')$ using colors from $\{1, \ldots, t\}$. For $t = 2$, the proof is trivial since the only minimal b-2-atom is $P_2$ and we can easily extend the coloring to $N(P_2)$. Thus we can suppose that $t \geq 3$. If $u$ has at most $t - 1$ neighbors in $N[T']$. Thus we can suppose that $t \geq 3$. If $u$ has at most $t - 1$ neighbors in $N[T']$,
then we can extend the coloring. Thus, we suppose that $u$ has $t$ neighbors in $N[T']$. In this case, $T'$ has $t$ connected components which are all stars. Each vertex of $N(u) \cap N[T']$ is either a vertex of a connected component of $T'$ or a vertex in $N(T')$ which is adjacent to one vertex of $V(T')$. In these two cases the vertices of $N(u) \cap N[T']$ should be in or be adjacent to vertices of disjoint connected components of $T'$. Thus the vertices of $N(u) \cap N(T')$ have at most two neighbors in $N[T']$: the vertex $u$ and another vertex of $T'$ (otherwise, there is a cycle in $T$). We begin by giving a color from $\{1, \ldots, t\}$ to the vertices of $N(T') \setminus \{u\}$. The vertex $u$ can not be adjacent to all vertices of $C(T')$ since otherwise it would contradict $t = \varphi_r(T)$. Let $v \in N[T'] \setminus C(T')$ be a neighbor of $u$. If $v \in N(T')$, then $v$ has at most two neighbors in $N[T']$ and $v$ can be recolored in order to color $u$. If all neighbors of $u$ are in $T'$, then $v \in N(c_i)$, for $i \in \{1, \ldots, t\}$ and we can exchange the color of $v$ with the color of a vertex $w \in N(c_i) \setminus \{v\}$ in order to color $u$ (since $t \geq 3$, $N(c_i) \setminus \{v\}$ is not empty). Finally, the vertices of $N(u) \cap N(T')$ can be recolored if we have obtained an improper coloring by recoloring $w$.

The girth of a graph $G$ is the length of a smallest cycle in $G$. We finish this paper by proving that when a graph $G$ has sufficiently large girth, we have $\varphi(G) = \varphi_r(G)$, thus extending Proposition 6.2.

**Theorem 6.3.** Let $G$ be a graph with girth $g$ and $\varphi_r(G) \geq 3$. If $g \geq 7$, then $\varphi(G) = \varphi_r(G)$.

**Proof.** Let $t = \varphi_r(G)$. By Proposition 3.9, it suffices to prove that every minimal b-$t$-atom is feasible to have $\varphi(G) = \varphi_r(G)$. Let $A_t$ be a minimal b-$t$-atom. Our proof consists in extending the coloring of $A_t$ induced by $D_1, \ldots, D_t$ to $N(A_t)$ using colors from $\{1, \ldots, t\}$. Thus, we consider that the vertices of $A_t$ are already colored.

For a vertex $u \in N(A_t)$, we denote by $I_u$ the set $\{t \in \{1, \ldots, t\} \mid \exists v \in N(u) \cap N[c_i]\}$. For a vertex $u \in V(A_t)$, we denote by $c^u$ a neighbor of $u$ in $C(A_t)$ if $u \notin C(A_t)$ or the vertex $u$ itself if $u \in C(A_t)$. Finally, we denote by $N(A_t)$ the set of vertices $V(A_t) \cup N(A_t)$. In the different cases, when we describe a cycle of length at most $k$ by $u_1, \ldots, u_k$, it is assumed that, depending the configuration, consecutive symbols can denote the same vertex. In this proof, any considered vertex is supposed to be in $N(A_t)$. We begin by proving the following properties:

- i) No vertex of $N(A_t)$ is adjacent to two vertices of $N[c_i]$, for $1 \leq i \leq t$;
- ii) If $u, v \in N(A_t)$ and $i \in I_u \cap I_v$, then $u$ and $v$ are not adjacent and have no common neighbor in $N(A_t) - c_i$;
- iii) If $u, v \in N[c_i]$ and $u', v' \in N[c_j]$, $u \neq v$, $u' \neq v'$, for some $i$ and $j$, $1 \leq i < j \leq t$, then the subgraph induced by $\{u, v, u', v'\}$ contains at most one edge.

- i) If $u$ is adjacent to two vertices of $N[c_i]$, for some $i$, $1 \leq i \leq t$, then $u$ is in a cycle of length at most 4. This cycle contains $u$, $c_i$ and one or two vertices of $N[c_i]$.
ii) If \( u \) and \( v \) are adjacent or have a common neighbor, then \( u \) and \( v \) belong to a cycle of length at most 6. This cycle contains \( u, v \), vertices of \( N[c_i] \) and possibly the common neighbor of \( u \) and \( v \) in \( N(A_t) - c_i \), for \( i \) an integer such that \( i \in I_c(u) \cap I_c(v) \).

iii) If the subgraph induced by \{\( u, v, u', v' \)\} contains at least 2 edges, then there is a cycle of length at most 6 in \( G \). This cycle is \( u-v-c_i \) if \( u \) and \( v \) are adjacent, \( u'-v'-c_j \) if \( u' \) and \( v' \) are adjacent or the cycle \( u-c_i-v-u'-v'-c_j \), otherwise.

We are going to prove that either each vertex \( u \in N(A_t) \) can be colored with colors from \{\( 1, \ldots, t \)\} or the graph \( G \) contains a \( b-(t+1) \)-atom (which contradicts \( \varphi_e(G) = t \)). By properties i) and ii), any vertex of \( N(A_t) \) has at most \( t \) neighbors in \( N[A_t] \). Hence we may suppose that any vertex \( u \in N(A_t) \) with less than \( t \) neighbors in \( N[A_t] \) is already colored and only consider vertices of \( N(A_t) \) with \( t \) neighbors in \( N[A_t] \). For a vertex \( u \in N[A_t] \), a color \( i \) is said to be available for \( u \) if no vertex has color \( i \) in \( N(u) \cap N[A_t] \) (and therefore, \( u \) has no available color if the colors \( 1, \ldots, t \) are not available for \( u \)). Let \( N_u(A_t) \) be the set of vertices in \( N(A_t) \) with no available colors.

We define the following three sets:

- \( N_1 = \{ u \in N(A_t) | N(u) \cap (V(A_t) \setminus C(A_t)) \neq \emptyset, N(u) \cap N(A_t) = \emptyset \} \);
- \( N_2 = \{ u \in N(A_t) | N(u) \cap (V(A_t) \setminus C(A_t)) = \emptyset, N(u) \cap N(A_t) \neq \emptyset \} \);
- \( N_3 = \{ u \in N(A_t) | N(u) \cap (V(A_t) \setminus C(A_t)) = \emptyset \} \).

We can remark that \( N_1 \cup N_2 \cup N_3 = N(A_t) \).

In the remainder of the proof we will first consider the vertices of \( N_1 \); secondly the vertices of \( N_2 \); and finally the vertices of \( N_3 \).

**Case 1:** vertices of \( N_1 \).

Let \( u \) be a vertex of \( N_1 \). We recall that, by the above assumption, \( u \) has exactly \( t \) neighbors in \( A_t \). Moreover, by Property i), \( |I_c(u)| = t \). Let \( c_i \in C(A_t) \). We denote by \( A^*_i \) the vertices of \( N[c_i] \) which have a neighbor in \( N_u(A_t) \). Notice that a vertex \( v \in A^*_i \) cannot have a neighbor \( x \) in \( V(A_t) \setminus \{c_i\} \) since otherwise it would create a cycle \( v-x-c_i-v'-u \), where the neighbor of \( v \) in \( N_1 \) and \( N_u(A_t) \) and \( v' \) the neighbor of \( u \) in \( N[c_i] \). This cycle has length at most 5, contradicting \( g \geq 7 \). If for a vertex \( c_i \in C(A_t) \) we have \( |A^*_i| \geq 2 \), we exchange the colors of the vertices of \( A^*_i \) by doing a cyclic permutation of their colors. Afterwards, we obtain that some vertices of \( N_1 \cap N_u(A_t) \) have now an available color and we recolor them by any available color. Finally, we color the vertices of \( N_1 \), when possible, by any available color. Let \( N_{u*}(A_t) \) be the set of the remaining uncolored vertices of \( N_1 \). In the following subcases, we recolor at most once the vertices of \( N[c_i] \), for \( i \in \{1, \ldots, t\} \), since any two vertices of \( N_{u*}(A_t) \) cannot both have neighbors in \( N(c_i) \).
By considering that $N_*(A_t) \neq \emptyset$ (or else we have nothing more to do in Case 1)), we can suppose that for every two integers $i, j$, $1 \leq i \neq j \leq t$, we have $N[v_i] \cap N[v_j] = \emptyset$. Otherwise, if there exists a vertex $u \in N_*(A_t)$ and a vertex $w \in N[v_i] \cap N[v_j]$, there is a cycle $u-v_i-w-v_j-u'$ of length at most 6, for $v$ a neighbor of $u$ in $N[v_i]$ and $v'$ a neighbor of $u$ in $N[v_j]$. Thus, we obtain that if $N_*(A_t) \neq \emptyset$, then every vertex $c_i \in C(A_t)$ has only one neighbor of color $j$, for $1 \leq i \neq j \leq t$, since otherwise it would contradict the minimality of $A_t$ (by removing one vertex of color $j$).

We then consider the two following subcases, for $u \in N_*(A_t)$.

**Subcase 1.1:** $u$ has exactly one neighbor in $V(A_t) \setminus C(A_t)$.

Let $v'$ be the neighbor of $u$ in $V(A_t) \setminus C(A_t)$ and let $c'$ be the color of $v'$. Notice that no vertex $x$ from $N[c']$ has a neighbor $y$ in $V(A_t) \setminus N[c']$, since otherwise it would create a cycle $u-v'-x-y-c''$ of length at most 6. Consequently, we can exchange the color of $v'$ with the color of one vertex from $N[c']$ and color $u$ by $c'$.

**Subcase 1.2:** $u$ has more than one neighbor in $V(A_t) \setminus C(A_t)$.

Let $v_1$ and $v_2$ be two neighbors of $u$ in $V(A_t) \setminus C(A_t)$. Let $c'$ be the color of $v_1$ and let $c''$ be the color of $v_2$.

If $v_1$ has a neighbor $x \in V(A_t) \setminus N[c''1]$, then there exists a cycle $u-v_1-x-v''-v'$ in $G$, with $v''$ a neighbor of $c''$ in $N(u)$ (in the case $c''$ is not a neighbor of $u$). Similarly if $v_2$ has a neighbor in $V(A_t) \setminus N[c''2]$, then there is a cycle of length at most 5 in $G$. Consequently, we can suppose that $v_1$ has no neighbor in $V(A_t) \setminus N[c''1]$ and that $v_2$ has no neighbor in $V(A_t) \setminus N[c''2]$. If there exists a vertex of $N(c''1) \setminus \{v_1\}$ with no neighbor of color $c'$, then
we exchange the color of $v_1$ with the color of this vertex and color $u$ by $c'$. If there exists a vertex of $N(e^{v_2}) \setminus \{v_2\}$ with no neighbor of color $c''$, then we exchange the color of $v_2$ with the color of this vertex and color $u$ by $c''$. Thus, we may suppose that every vertex $w$ of $N(e^{v_1}) \setminus \{v_1\}$ (of $N(e^{v_3}) \setminus \{v_2\}$, respectively) has a neighbor $\overline{v}$ of color $c'$ ($c''$, respectively) in $V(A_i)$. We consider three subcases in order to color to $u$.

**Subcase 1.2.1:** the vertices $v_1$ and $e^{v_2}$ have the same color and the vertices $v_2$ and $e^{v_1}$ have the same color.

Notice that no vertex $w \in N(e^{v_1})$ is adjacent to $e^{v_2}$ since otherwise $u$-vertex $v_1-e^{v_1}-w-e^{v_2}-v_2$ would be a cycle of length at most 6 in $G$. For the same reason, no vertex $w \in N(e^{v_2})$ is adjacent to $e^{v_1}$. Thus, by Property iii), no vertex $w \in N[e^{v_1}] \cup N[e^{v_2}]$ has a neighbor $x \in V(A_i) \setminus (N[e^{v_1}] \cup N[e^{v_2}] \cup \{\overline{v}\})$, since there exists a vertex $y \in N(e^{w})$ with neighbor $\overline{v} \in N(e^{v_2})$. There could exist two adjacent vertices $w$ and $w'$ with $w \in N(e^{v_1})$ and $w' \in N(e^{v_2})$. However, the vertex $w'$ has no neighbor of color $c''$ in $A_i$ since $w'$ and $v_2$ can not be adjacent and there does not exist a second vertex of color $c''$ in $N(e^{v_2})$. Consequently, we can exchange the color of $v_1$ with the color of $v_2$, the color of $e^{v_2}$ with the color of $e^{v_1}$ and afterward we can exchange the color of one vertex from $N(e^{v_1}) \setminus \{v_1\}$ with the color of $v_1$ and color $u$ by $c''$. The top of Figure 5 illustrates this recoloring process on a minimal b-4-atom fulfilling the hypothesis of Subcase 1.2.1.

**Subcase 1.2.2:** the vertices $v_1$ and $e^{v_2}$ do not have the same color.

Let $i$ be the color of $e^{v_1}$ and $j$ be the color of $e^{v_2}$. In this case, we exchange the color of $e^{v_2}$ with the color of $e^{v_1}$ and the color of the vertex $w$ of color $j$ in $N(e^{v_1})$ with the color of the vertex $w'$ of color $i$ in $N(e^{v_2})$. For this, we have to suppose that $w$ is not adjacent to a vertex of color $i$ and that $w'$ is not adjacent to a vertex of color $j$. For $t \geq 4$, such vertices $w$ and $w'$ exist since at most one vertex of $N(e^{v_1})$ has a neighbor of color $j$ (otherwise, it would contradict Property iii) since every vertex of $N(e^{v_1}) \setminus \{v_1\}$ has already a neighbor in $V(A_i)$ of color $c'$) and at most one vertex of $N(e^{v_2})$ has a neighbor of color $i$. If $t = 3$, then the only (up to isomorphism) b-3-atom with a coloring fulfilling all these hypothesis (up to color permutation) is illustrated at the bottom of Figure 5, along with the recoloring process. In this b-3-atom, no more edge can be added (otherwise, it would create a cycle of length at most 6).

**Subcase 1.2.3:** the vertices $v_2$ and $e^{v_1}$ do not have the same color.

We proceed as for the previous subcase by considering $v_2$ instead of $v_1$ and $e^{v_1}$ instead of $e^{v_2}$.

**Case 2:** vertices of $N_2$.

Since each pair of adjacent vertices $u, v \in N(A_i)$ satisfies Property ii), we obtain that $I_c(u) \cap I_c(v) = \emptyset$. We color each vertex $u \in N_2$ by a color $i \in I_c(u)$ such that $u$ and $c_i$ are not adjacent.
Case 3: vertices of $N_3$.

Notice that, by definition, a vertex of $C(A_t)$ has no available color. Let $u \in N_3$. We begin by coloring $u$ with any available color if it has some. If $u$ has no available color, there could exist a color $i$ such that every vertex of $N(u)$ with color $i$ has an available color (these vertices should be in $N(A_t)$). If such color $i$ exists, we recolor these vertices of color $i$ by any available color and give color $i$ to $u$. If such color $i$ does not exist, then the set of vertices at distance at most 2 from $u$ induces a b-$(t+1)$-atom with center $N[u]$. It can be noted that the recolored vertices are in $N(A_t)$ since $N(u) \cap V(A_t) \subseteq C(A_t)$.

We finish this proof by illustrating that the obtained coloring is a b-$t$-coloring of $N[A_t]$. In case 1, we have modified the coloring of $A_t$. However, since we have exchanged the colors of well-chosen vertices in order that every vertex of $C(A_t)$ still has neighbor of every color from $\{1, \ldots, t\}$ except its own color, this coloring remains a b-$t$-coloring. In case 3, we have only changed the color of vertices from $N(A_t)$.

We think that the previous theorem can be useful to determine the family of graphs of girth at least 7 satisfying $\varphi(G) = m(G)$. It has already been proven that graphs of girth at least 7 have b-chromatic number at least $m(G) - 1$ [7].

**Corollary 6.4.** Let $G$ be a graph of girth at least 7 and of order $n$ and let $t$ be an integer. There exists an algorithm in time $O(n^{t^2})$ to determine if $\varphi(G) \geq t$.

7 Open questions

We conclude this article by listing few open questions.

1. For which family of graphs are the b-relaxed number and the b-chromatic number equal?

2. Does there exist an easy characterization of feasible b-$t$-atoms?

3. Does there exist an FPT algorithm, with parameter $t$, to determine if $\varphi(G) \geq t$?

References


