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Extended Abstract

In this extended abstract, we only consider connected graphs. Let $k \geq 2$ be an integer and $T_1, \ldots, T_k$ be spanning trees in a graph G. A vertex is said to be an *inner vertex* in a tree T if it has degree at least 2 in T. We denote by $I(T)$ the set of inner vertices of tree T. The spanning trees $T_1, \ldots, T_k$ are *completely independent spanning trees* if any vertex from G is an inner vertex in at most one tree among $T_1, \ldots, T_k$ and the trees $T_1, \ldots, T_k$ are pairwise edge-disjoint.

Completely independent spanning trees were introduced by Hasunuma [4] and then have been studied on different classes of graphs, such as underlying graphs of line graphs [4], maximal planar graphs [5], Cartesian product of two cycles [6] and k-trees [10]. Moreover, determining if there exist two completely independent spanning trees in a graph G is a NP-hard problem [5]. Recently, sufficient conditions inspired by the sufficient conditions for hamiltonicity have been determined in order to guarantee the existence of several completely independent spanning trees: Dirac's condition [1] and Ore's condition [2]. Moreover, Dirac's condition has been generalized to more than two trees [7].

In this extended abstract, we introduce (i, j)-disjoint spanning trees:

Definition 0.1 *Let $k \geq 2$ be an integer and $T_1, \ldots, T_k$ be spanning trees in a graph G. We let $I(T_1, \ldots, T_k) = \{u \in V(G) | \exists \ell, \ell' \ u \in I(T_\ell) \cap I(T_{\ell'}),\ 1 \leq \ell < \ell' \leq k\}$ be the set of vertices which are inner vertices in at least two spanning trees among $T_1, \ldots, T_k$, and we let $E(T_1, \ldots, T_k) = \{e \in E(G) | \exists \ell, \ell',\ 1 \leq \ell < \ell' \leq k,\ e \in E(T_\ell) \cap E(T_{\ell'})\}$ be the set of edges which belong to at least two spanning trees among $T_1, \ldots, T_k$. The spanning trees $T_1, \ldots, T_k$ are* (i, j)-disjoint *for two positive integers i and j, if the two following conditions are satisfied:*

i) $|I(T_1, \ldots, T_k)| \leq i$;

ii) $|E(T_1, \ldots, T_k)| \leq j$.

The sets $D_1, \ldots D_k$ in a graph G are *disjoint connected dominating* sets if they are pairwise disjoint and dominating. Moreover, if $|\cup_{1 \leq i < j \leq k} D_i \cap D_j| \leq \ell$ we say that $D_1, \ldots D_k$ are *ℓ-rooted connected dominating sets.* Other works on disjoint spanning trees are about *disjoint connected dominating sets* (the disjoint connected dominating sets can be used to provide the inner vertices of $(0, E(G))$-disjoint spanning trees). The maximum number of disjoint connected dominating sets in a graph G is the *connected domatic number* [12]. An interesting result about connected domatic number concerns planar graphs, for which Hartnell and Rall have proven that, except K_4 (which has connected domatic number 4), their connected domatic number is bounded by 3 [3]. The problem of constructing a connected dominating set is often motivated by wireless ad-hoc networks [11]. Connected dominating sets are used to create a virtual backbone or spine of a wireless ad-hoc network.

By $*$ we denote a large enough integer, i.e. an integer larger than $\max(|E(G)|, |V(G)|)$, for a graph G. Remark that $(0, 0)$-disjoint spanning trees are completely independent spanning trees and that $(*, 0)$-disjoint spanning trees are edge-disjoint spanning trees. Also, $(0, *)$-disjoint spanning trees are related to connected dominating sets. Hence, we call them trees induced by disjoint connected dominating sets. For the same reason than $(0, *)$-disjoint spanning trees, $(\ell, *)$-disjoint spanning trees are trees induced by ℓ-rooted connected dominating sets. In the following sections, we illustrate that (i, j)-disjoint spanning trees provide some nuances between the existence of disjoint connected dominating sets and of completely independent spanning trees.

1 Characterizations in terms of partitions

We introduce a definition which is a generalization of CIST-partition introduced by Araki [1]. Let V_1 and V_2 be two disjoint subsets of vertices of a graph G. By $B(V_1, V_2)$ we denote the bipartite graph with vertex set $V_1 \cup V_2$ and edge set $\{uv \in E(G)|\ u \in V_1,\ v \in V_2\}$. An *$\ell$-CIST-partition* of a graph G into k sets is a partition of $V(G)$ into k sets of vertices $V_1, \ldots, V_k$ such that:

i) $G[V_i]$ is connected, for each integer i, $1 \le i \le k$;

ii) $B(V_i, V_j)$ contains no isolated vertex, for every two integers i, j, $1 \le i < j \le k$;

iii) $\sum_{1 \le i < j \le k} c_{i,j} \le \ell$, where $c_{i,j}$ is the number of connected component which are trees in $B(V_i, V_j)$, $1 \le i < j \le k$.

Theorem 1.1 *Let G be a graph. There exist k $(0,\ell)$-disjoint spanning trees $T_1, \ldots, T_k$ in G if and only if G has an ℓ-CIST-partition into k sets.*

Proof. Suppose G has an ℓ-CIST-partition into k sets $V_1,\ldots,V_k$. We are going to build $(0,\ell)$-disjoint spanning trees $T_1, \ldots, T_k$. We begin by setting $I(T_i) = V_i$ for each integer i, $1 \le i \le k$. For now, we suppose that $E(T_i)$ is empty and we progressively add edges in $E(T_i)$, for each integer i, $1 \le i \le k$, in order to obtain spanning trees of G. Since $G[V_i]$ is connected for each i, $1 \le i \le k$, we can add edges in $E(T_i)$ in order to form a tree with vertex set V_i, for each i.

Let i and j be two integers, $1 \le i < j \le k$, and let $D_{i,j}$ be a connected component of $B(V_i, V_j)$. We add edges in order to build a spanning tree restricted to $V_i \cup V(D_{i,j})$ and another spanning tree restricted to $V_j \cup V(D_{i,j})$ by considering two cases. Let u be a vertex of $D_{i,j} \cap V_i$. First, if $D_{i,j}$ is a tree, then we add one edge e of $D_{i,j}$ with extremity u in $E(T_1, \ldots, T_k)$. Let $D^d_{i,j}(u) = \{v \in V(D_{i,j})|\ d_{D_{i,j}}(u, v) = d\}$. We add the following edges to $E(T_1)$: $\{vv' \in E(D_{i,j})|\ v \in D^d_{i,j}(u),\ v' \in D^{d+1}_{i,j}(u),\ d \text{ is even}\}$ and the following edges to $E(T_2)$: $\{vv' \in E(D_{i,j})|\ v \in D^d_{i,j}(u),\ v' \in D^{d+1}_{i,j}(u),\ d \text{ is odd}\}$. Second, if $D_{i,j}$ is not a tree, then we suppose that u is in a cycle of $D_{i,j}$. Let e be an edge of this cycle incident with u and let $T_{i,j}$ be a spanning tree of $D_{i,j} - e$. We define $D^d_{i,j}(u)$ as follows: $\{v \in V(D_{i,j})|\ d_{T_{i,j}}(u, v) = d\}$. We add the following edges to $E(T_1)$: $\{vv' \in E(T_{i,j})|\ v \in D^d_{i,j}(u),\ v' \in D^{d+1}_{i,j}(u),\ d \text{ is even}\}$ and the following edges to $E(T_2)$: $\{vv' \in E(T_{i,j})|\ v \in D^d_{i,j}(u),\ v' \in D^{d+1}_{i,j}(u),\ d \text{ is odd}\} \cup \{e\}$. Note that $e \in E(T_1, \ldots, T_k)$. We repeat this process for every connected component of $B(V_i, V_j)$ and every integers i and j, $1 \le i < j \le k$. Since we have $\sum_{1 \le i < j \le k} c_{i,j} \le \ell$, the set $E(T_1, \ldots, T_k)$ contains at most ℓ edge. Therefore we obtain, by Property ii), k $(0,\ell)$-disjoint spanning trees.

Let us prove the converse of the previous implication. Suppose there exist k $(0,\ell)$-disjoint spanning trees $T_1, \ldots, T_k$ in G. The set $I(T_i)$, $1 \le i \le k$ induces a connected subgraph in G. We begin by setting $V_i = I(T_i)$, for each integer i, $1 \le i \le k$. If some vertices are inner vertices in no trees, we can add them to any set among $V_1, \ldots, V_k$ for which they have a neighbor. Thus, Property i) follows. Let i and j be two integers, $1 \le i < j \le k$. Suppose there exists one isolated vertex u in $B(V_i, V_j)$. Without loss of generality, suppose $u \in V_i$. We have a contradiction since u has no neighbor in $I(T_j)$ and T_j is supposed to be a spanning tree. Thus, Property ii) follows. Now suppose $\sum_{1 \le i < j \le k} c_{i,j} > \ell$. Let $D_{i,j}$ be a connected component which is a tree in $B(V_i, V_j)$ for some integers i and j and suppose that $D_{i,j}$ contains no edge from $E(T_1, \ldots, T_k)$. Since $D_{i,j}$ contains $|V(D_{i,j})| - 1$ edges, it is impossible that every vertex of $V(D_{i,j}) \cap V_i$ is adjacent to a vertex of $V(D_{i,j}) \cap V_j$ in T_j and every vertex of $V(D_{i,j}) \cap V_j$ is adjacent to a vertex of $V(D_{i,j}) \cap V_i$ in T_i, since it would require $|V(D_{i,j})|$ edges. Thus, for every two integers i and j and each connected component of $B(V_i, V_j)$ which is a tree, we need an edge in $E(T_1, \ldots, T_k)$. Thus, we obtain a contradiction since we obtain trees which are not $(0,\ell)$-disjoint spanning trees and Property iii) follows.

□

For a graph G and subset of vertices $A \subseteq V(G)$, let $N(A) = \{u \in V(G) \setminus A |\ uv \in E(G),\ v \in A\}$. In a similar way than Zelinka [12], we prove that the notion of ℓ-rooted connected dominating sets is equivalent to a notion of partition. An *ℓ-rooted partition* of a graph G into $k+1$ sets is a partition of $V(G)$ into $k+1$ sets of vertices $V_1, \dots, V_k, A$ such that:

i) $|A| \leq \ell$;

ii) $G[V_i \cup A]$ is connected, for each i, $1 \leq i \leq k$;

ii) $B(V_i, V_j) - N(A)$ contains no isolated vertex, for every i and j, $1 \leq i < j \leq k$.

Theorem 1.2 *Let G be a graph. There exist k ℓ-rooted connected dominating sets $D_1, \dots, D_k$ in G if and only if G has an ℓ-rooted partition into $k+1$ sets.*

Sketch of Proof. The proof is similar to the one of Theorem1.1.

2 Computational complexity and connectivity

We define the following decision problem:

> k-(i,j)-**DSP**
> **Instance** : A graph G.
> **Question**: Does there exist k (i,j)-disjoint spanning trees in G ?

Theorem 2.1 *Let i and j be non negative integers. The problem 2-(i,j)-DSP is a NP-complete problem for every pair of integers (i,j).*

Sketch of proof. The proof uses a reduction from 3-SAT similar to the reduction used by Hasunuma [5].

□

Moreover, since the presence of a k-cut in a graph G implies that there do not exist $k+1$ disjoint connected dominating set, it is natural to ask whether k-connected graph, for k sufficiently large, contains at least two (i,j)-disjoint spanning trees or not.

Theorem 2.2 *Let i, j and k be integers. For any positive integer k, there exist a k-connected graph which does not contain two (i,j)-disjoint spanning trees.*

Sketch of proof. The considered graphs are the same than the graphs introduced by Kriesell [9] (they are the incidence graphs of complete k-uniform hypergraphs). The proof consists in proving that the existence of two (i,j)-disjoint spanning trees implies that there exists a vertex u in this graph for which the vertices of $N(u)$ are all inner vertices of the same tree. Moreover, no vertex of $N(u)$ should be inner vertex of both trees. These facts imply that u cannot be in one of the two spanning trees and thus a contradiction.

□

3 Some simple classes of graphs

We finish this extended abstract by giving some results for square of graphs, complete graphs and square grids.

Theorem 3.1 *Let G be graph. There exists two $(0,1)$-disjoint spanning trees in G^2 and there do not exist two completely independent spanning trees in G^2 if and only if G is a tree from the family described by Araki [1].*

Sketch of proof. Let V_1 and V_2 be the set of vertices induced by a bipartition of a spanning tree of G. It easy to prove that V_1 and V_2 form a 1-CIST-partition of G. Moreover, Araki has characterized the trees containing two completely independent spanning trees. It suffice to prove that a connected graph G which is not a tree contains two completely independent spanning trees to complete the proof. That is the case since the set of vertices induced by a bipartition of a spanning tree of $G - \{e\}$ for e an edge of a cycle of G, is a 0-CIST-partition.

□

Remark that there are n disjoint connected dominating sets in K_n and that there are $\lfloor n/2 \rfloor$ completely independent spanning trees in K_n [10] and that there does not exist two disjoint connected dominating sets in a sufficiently large square grid [3]. We give the following intermediate result about (i, j)-disjoint spanning trees (the proof is not given):

Theorem 3.2 *Let n be an integer. There are at most $\lfloor n/2 \rfloor + \max(\lfloor \ell/(n-1) + 1_{odd}(n)/2 \rfloor, \lceil n/2 \rceil)$ $(0, \ell)$-disjoint spanning trees in K_n, where $1_{odd}(n) = 1$ if n is odd and 0 otherwise.*

Also, there exist two 1-rooted connected dominating sets in $P_{n_1} \square P_{n_2}$, for every $n_1 \geq 3$ and $n_2 \geq 3$.

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