

INFINITE ORBIT DEPTH AND LENGTH OF MELNIKOV FUNCTIONS

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ABSTRACT. In this paper we study polynomial Hamiltonian systems $dF = 0$ in the plane and their small perturbations: $dF + \epsilon\omega = 0$. The first nonzero Melnikov function $M_\mu = M_\mu(F, \gamma, \omega)$ of the Poincaré map along a loop γ of $dF = 0$ is given by an iterated integral [3]. In [7], we bounded the length of the iterated integral M_μ by a geometric number $k = k(F, \gamma)$ which we call *orbit depth*. We conjectured that the bound is optimal.

Here, we give a simple example of a Hamiltonian system F and its orbit γ having infinite orbit depth. If our conjecture is true, for this example there should exist deformations $dF + \epsilon\omega$ with arbitrary high length first nonzero Melnikov function M_μ along γ . We construct deformations $dF + \epsilon\omega = 0$ whose first nonzero Melnikov function M_μ is of length three and explain the difficulties in constructing deformations having high length first nonzero Melnikov functions M_μ .

1. INTRODUCTION AND MAIN RESULTS

This paper is motivated by two classical problems in the study of orbits of vector fields in the plane: the 16-th Hilbert problem and the center problem or rather their infinitesimal versions.

The Infinitesimal Hilbert 16-th problem asks for a bound on the number of limit cycles (i.e. isolated periodic orbits) created by a small polynomial deformation of a given degree of an integrable vector field in the plane.

The infinitesimal center problem asks for a characterization of polynomial deformations of an integrable system which preserve a family of loops.

In both problems one studies the Poincaré first return map (1.2) on a transversal. The first (possibly) nonzero term M_μ carries lots of information about the Poincaré map. Having an a priori estimate on its complexity would be very important for both infinitesimal problems. For the infinitesimal center problem, to have an a priori estimate on the length is similar to having an estimate on the stabilization index for Noetherian property.

It is known [2, 3], that when deforming a Hamiltonian vector field, the first nonzero term M_μ is an iterated integral of length not exceeding its order μ . However, the order μ in general depends on the deformation. In [7], we gave a bound on the length of M_μ by a geometric number *orbit depth* k which is independent on the deformation. We showed that in different cases this bound is optimal and we conjectured that it is so in general.

Date: May 13, 2018.

1991 Mathematics Subject Classification. 34C07 (primary), 34C05, 34C08 (secondary).

Key words and phrases. Iterated integrals, Center problem.

This research was supported by the ISRAEL SCIENCE FOUNDATION (grant No. 1167/17), UNAM PREI Dgapa, Unidad Mixta Internacional Laboratorio Solomon Lefschetz (LASOL), FONCICYT, Papiit Dgapa UNAM IN106217, ECOS Nord-Conacyt 249542 and Conacyt 291231.

In this paper we give an example where this bound is infinite. We believe that in the example one can construct deformations whose first nonzero Melnikov function M_μ is of arbitrarily high length. In that direction we construct for our example deformations having first nonzero Melnikov function M_μ of length 3 and show the difficulties in constructing deformations with higher length.

Remark 1.1.

- (i) Our example answers negatively a question asked by Gavrilov and Iliev in [4].
- (ii) Our example shows the complexity of both infinitesimal problems.

Let us be more precise. Let $F \in \mathbb{C}[x, y]$ be a polynomial and let $\gamma \in \pi_1(F^{-1}(t))$ be a loop for t a regular value of F . Consider a small polynomial deformation

$$dF + \epsilon\omega = 0, \quad (1.1)$$

of the Hamiltonian $dF = 0$. Let τ be a transversal section to γ at a point p_0 , parametrized by the values t of F . Denote by P_γ the Poincaré return map (holonomy) of (1.1) along γ . Then

$$P_\gamma(t) = t + \epsilon^\mu M_{\gamma, \mu}(t) + o(\epsilon^\mu). \quad (1.2)$$

If the Poincaré map is not the identity map, we assume that M_μ is nonzero and call it the *first non-zero Melnikov function along γ of the deformation* (1.1).

By the Poincaré-Pontryagin criterion, the first order Melnikov function M_1 is given by an Abelian integral,

$$M_{\gamma, 1}(t) = \int_\gamma \omega. \quad (1.3)$$

More generally, $M_{\gamma, \mu}(t)$ is given as a linear combination of iterated integrals of length at most μ , see [2, 3]. However, this bound in general is not optimal. For instance, for generic F and any loop γ and any deformation ω , the first non-zero Melnikov functions $M_{\gamma, \mu}(t)$ is given by an Abelian integral (i.e. is an iterated integral of length 1), irrespective of its order μ . This follows from [6, 2], see [7]. For other examples see [7], as well as papers cited there. Moreover, the bound μ , for the length of $M_{\gamma, \mu}$ depends on the deformation (1.1).

In [4] a sufficient condition under which the first nonzero Melnikov function $M_{\gamma, \mu}(t)$ is an Abelian integral is formulated. We generalized this condition in [7]:

Let Σ be the set of atypical values of F , see [5], and let $t \notin \Sigma$ be some regular value of F . Denote $\Gamma_t = \{F^{-1}(t)\}$. The fundamental group $\pi_1(\mathbb{C} \setminus \Sigma, t)$ acts on the fundamental group $\pi_1(\Gamma_t, p_0)$ as follows. For each generator a_j of $\pi_1(\mathbb{C} \setminus \Sigma, t)$ corresponding to a closed curve $a_j(s) \subset \mathbb{C} \setminus \Sigma$, choose its lifting $\tilde{a}_j(s)$, i.e. a loop $\tilde{a}_j(s) \subset F^{-1}(\mathbb{C} \setminus \Sigma)$ such that $F(\tilde{a}_j(s)) = a_j(s)$ and $\tilde{a}_j(0) = \tilde{a}_j(1) = p_0$. Then, by Ehresmann's fibration theorem, the fundamental groups $\pi_1(F^{-1}(a_j(s)), \tilde{a}_j(s))$ and $\pi_1(F^{-1}(a_j(s')), \tilde{a}_j(s'))$ are canonically isomorphic for sufficiently close s, s' . This defines an automorphism $Mon(a_j)$ of $\pi_1(\Gamma_t, p_0)$ and the representation $Mon : \pi_1(\mathbb{C} \setminus \Sigma, t) \rightarrow Aut(\pi_1(\Gamma_t, p_0))$. This representation depends on the choice of the liftings \tilde{a}_j , and different choices of liftings change $Mon(a_j)$ to conjugate automorphisms $\sigma_j^{-1} Mon(a_j) \sigma_j$, $\sigma_j \in \pi_1(\Gamma_t, p_0)$. We fix some choice of \tilde{a}_j .

Definition 1.2 (see [4, 7]). Let \mathcal{O} be the smallest normal subgroup of $\pi_1(\Gamma_t, p_0)$ containing the orbit of $\gamma \in \pi_1(\Gamma_t, p_0)$ under the action of $Mon(\pi_1(\mathbb{C} \setminus \Sigma, t))$. Denote $K = [\mathcal{O}, \pi_1(\Gamma_t, p_0)]$ and let $H_1(\mathcal{O}) = \mathcal{O}/K$.

Remark 1.3. Note that \mathcal{O} , K and $H_1(\mathcal{O})$ are independent on the particular choice of \tilde{a}_j . Moreover, $H_1(\mathcal{O})$ is canonically isomorphic for different choices of p_0 in the following sense: the natural isomorphism between $\pi_1(\Gamma_t, p_0)$ and $\pi_1(\Gamma_t, p'_0)$ defined by a path joining p_0 and p'_0 descends to an isomorphism of the corresponding $H_1(\mathcal{O})$, and this isomorphism is independent on the choice of this path.

In what follows, we denote $\pi_1(\Gamma_t, p_0)$ by π_1 . The *lower central sequence* of π_1 is defined as:

$$\pi_1 = L_1 \supset L_2 = [L_1, \pi_1] \supset \cdots \supset L_{i+1} = [L_i, \pi_1] \supset \cdots \quad (1.4)$$

There is a natural homomorphism $\iota_1 : H_1(\mathcal{O}) \rightarrow H_1(\Gamma_t, \mathbb{C})$, and let $\mathcal{O}_1 = \iota_1(H_1(\mathcal{O})) = \frac{\mathcal{O}L_2}{L_2} \otimes \mathbb{C}$. In general, ι_1 is neither surjective nor injective. In [4] it is shown that if ι_1 is injective then $M_\mu(t)$ is an Abelian integral.

In [7], we defined the *orbit depth* $k = k(F, \gamma)$,

Definition 1.4. Given a polynomial $F \in \mathbb{C}[x, y]$ and a loop $\gamma \in \pi_1$ as above, the orbit depth $k = k(F, \gamma)$ is defined as

$$k = \sup \{j \geq 1 \mid \mathcal{O} \cap L_j \not\subseteq K\} \subset \mathbb{N} \cup \{+\infty\}. \quad (1.5)$$

We say that an element $v \in \mathcal{O}$ is of depth j if it belongs to L_j and its class in $H_1(\mathcal{O})$ is nonzero. Orbit depth is $k < \infty$ if k is the highest depth of elements in \mathcal{O} , and it is infinite if there are elements of \mathcal{O} of arbitrary high depth.

In [7, Theorem 1.7]) we proved that the orbit depth $k = k(F, \gamma)$ bounds the length of iterated integrals representing the first nonzero Melnikov function $M_{\gamma, \mu}$ of small deformations (1.1).

We conjectured that it was an optimal bound for the length of the first nonzero Melnikov function $M_{\gamma, \mu}$ along γ of deformations of $dF = 0$. We hence believe that for a Hamiltonian system $dF = 0$ and a loop $\gamma \in \pi_1(F^{-1}(t))$ of infinite orbit depth there exist polynomial deformations (1.1) such that the first non-zero Melnikov function $M_{\gamma, \mu}$ is an iterated integral of arbitrary high length.

Theorem 1.5. *There exists a polynomial function $F \in \mathbb{R}[x, y]$ and a loop $\gamma \in \pi_1(F^{-1}(t))$ such that the orbit depth k of γ is infinite.*

Such an example is given by

$$F(x, y) = (x^2 - 1)(y^2 - 1), \quad (1.6)$$

and the loop $\gamma \subset \{F = t\}$ given by the real cycle vanishing at $(0, 0)$ along the path $(0, t) \subset \mathbb{R}$, for $t \in (0, 1)$ (see Figure 1).

Our theorem also answers negatively to the question if $\dim H_1(\mathcal{O}) \leq \dim H_1(F^{-1}(t_0))$, which was raised as part of open question (1) in [4].

We also prove

Theorem 1.6. *There exist a rational deformation $dF + \epsilon\omega$ of F given by (1.6) such that the first nonzero Melnikov function $M_{\gamma, \mu}$ of the deformation (1.1) is an iterated integral of length 3. An example of such deformation is a form ω of type*

$$\omega = a_1(F) \frac{dx}{x+1} + a_2(F) \frac{dy}{y-1} + a_3(F) \frac{dx}{x-1}, \quad (1.7)$$

with $a_1(t) = t^2 + 2t$, $a_2(t) = t$ and $a_3(t) = t^2 + t$.

If $M_{\gamma, 2} = M_{\gamma, 3} \equiv 0$ for deformation (1.1) with ω as (1.7), then the deformation is integrable.

We conclude that one needs a richer set of deformations to get an example of a perturbation with first nonzero Melnikov function $M_{\gamma,\mu}$ of length ≥ 4 .

One of the principal tools of the proof is Proposition 3.4, establishing connection between Poincaré return maps of paths on Γ_t and the vector fields on the transversal τ whose flows give these maps.

2. EXAMPLE WITH INFINITE ORBIT DEPTH

We consider the polynomial $F(x, y) = (x^2 - 1)(y^2 - 1)$. The critical values of F are 0 and 1, and the critical points are $(\pm 1, \pm 1)$ on $\{F = 0\}$ and $(0, 0)$ at $\{F = 1\}$. Our goal in this section is to show that the orbit depth of the real cycle γ vanishing at the critical point $(0, 0)$ is infinity.

The normalizations Γ_t of complexifications of non-singular level curves $\{F = t\}$, $t \neq 0, 1$, are torii with 4 points removed. The fundamental group $\pi_1(\Gamma_t, p_0)$ is a free group generated by loops $\gamma, \delta_0, \delta_1, \delta_2, \delta_3$, where δ_i are loops vanishing at $(\pm 1, \pm 1)$.

To be more precise, we take $0 < t \ll 1$, choose p_0 close to the edge $\{x = -1\}$ of the square, and denote $\delta_0, \delta_1, \delta_2, \delta_3$ the geometric loops vanishing at $(-1, -1)$, $(1, -1)$, $(1, 1)$ and $(-1, 1)$ correspondingly, see the figure at [7]: we take a meridian of the cylinder which is $\{F = t\}$ near the corresponding singular point, with base point on γ , and then pull the base point clockwise along γ to p_0 . We orient γ counterclockwise, and orient δ_i in such a way that the intersection numbers (γ, δ_i) are all equal to one.

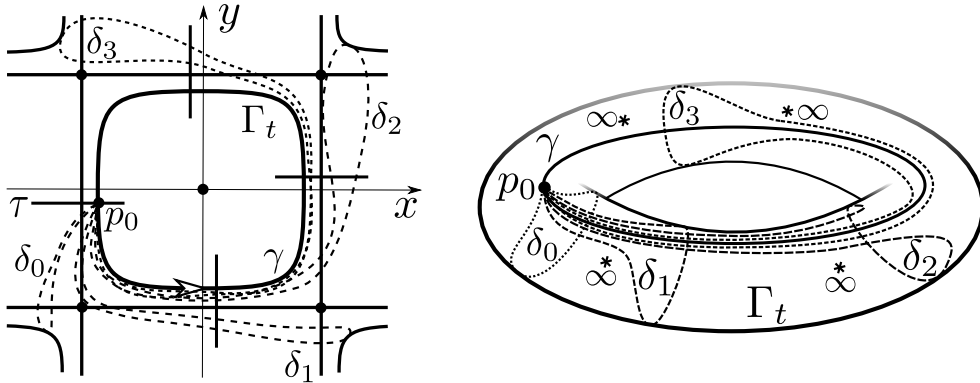


FIGURE 1. Generators of $\pi_1(\Gamma_t, p_0)$.

The atypical values of H are exactly its critical values 0, 1. Therefore, the action of the monodromy of the foliation on the fundamental group of Γ_t is generated by two automorphisms Mon_0 and Mon_1 of $\pi_1(\Gamma_t, p_0)$ corresponding to the loops going around the critical values 0 and 1 correspondingly, as described above.

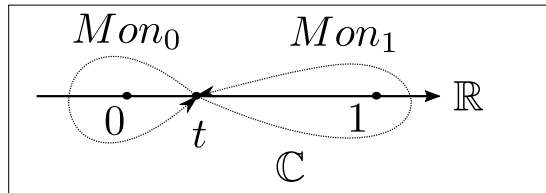


FIGURE 2. Generators of $\pi_1(\mathbb{C} \setminus \{0, 1\}, t)$ corresponding to Mon_0, Mon_1 .

Lemma 2.1. Denote $\delta = \delta_0\delta_1\delta_2\delta_3$. The monodromy operators $Mon_{0,1}$ are

$$Mon_1 = \{\gamma \mapsto \gamma, \delta_i \mapsto \gamma\delta_i\} \quad (2.1)$$

$$\begin{aligned} Mon_0 = \{ & \gamma \mapsto \delta\gamma, \delta_0 \mapsto \delta_0, \\ & \delta_1 \mapsto \delta_0\delta_1\delta_0^{-1}, \\ & \delta_2 \mapsto \delta_0\delta_1\delta_2\delta_1^{-1}\delta_0^{-1}, \\ & \delta_3 \mapsto \delta_0\delta_1\delta_2\delta_3\delta_2^{-1}\delta_1^{-1}\delta_0^{-1}\} \end{aligned} \quad (2.2)$$

Proof. These formulas follow from standard homotopical computations proving the Picard-Lefschetz formula, see e.g. [1]. From the local topology in a neighborhood of the center critical point, we have $Mon_1\gamma = \gamma$, and $Mon_1\delta_i = \gamma\delta_i$, since the intersection number between γ and δ_i is one, and γ is the cycle vanishing at the center critical value when the regular value tends to 1. For the monodromy around the critical value 0, we divide the real cycle γ into pieces $\gamma = \rho_0\rho_1\rho_2\rho_3$, where ρ_i goes from a point x_i in $\gamma \cap U_i$ to a point x_{i+1} in $\gamma \cap U_{i+1}$, where U_j is a neighborhood of the critical point at which δ_j vanish, with x_0 and x_4 equal to the chosen initial point p_0 . Let δ_i° be the generator of $\pi_1(F^{-1}(t) \cap U_i, x_i)$. From Picard-Lefschetz formula, locally at the neighborhood U_i we have $Mon_0(\rho_i) = \delta_i^{\circ}\rho_i$, and $Mon_0\delta_i^{\circ} = \delta_i^{\circ}$. Notice that $\delta_i = \rho_0 \cdots \rho_i \delta_i^{\circ} \rho_i^{-1} \cdots \rho_0^{-1}$. Then, applying the local monodromy at each saddle critical point we get the result. \square

2.1. Chipping out Homology and Mon_1 . Note that $\delta = Mon_0(\gamma)\gamma^{-1}$ is in \mathcal{O} , and that γ, δ span the orbit of γ in $H_1(\Gamma_t)$. From the exact sequence

$$0 \rightarrow (\mathcal{O} \cap L_2)/K \rightarrow \mathcal{O}/K \rightarrow \mathcal{O}_1 \rightarrow 0,$$

where $\mathcal{O}_1 = \langle \gamma, \delta \rangle \subset H_1(\Gamma_t)$, it is clear that the next step is to consider the action of the monodromy on $L_2 = [\pi_1, \pi_1]$. It turns out that, up to subgroup generated by γ and δ , the action of Mon_1 on L_2 is trivial, thus allowing to disregard Mon_1 .

Let Γ be the normal subgroup of π_1 generated by γ, δ . Evidently, $\Gamma \subset \mathcal{O}$ and $[\Gamma, \pi_1] \subset K$ is a normal subgroup of π_1 generated by commutators $[\gamma, c], [\delta, c], c \in \pi_1$.

The group $(\mathcal{O} \cap L_2)/K$ is a subgroup of L_2/K , which is a factor of $L_2/[\Gamma, \pi_1]$. The latter is isomorphic to the commutator $G_2 = [G, G]$ of the free group G generated by $\delta_1, \delta_2, \delta_3$.

Lemma 2.2. Mon_1 preserves both L_2 and Γ . The induced action of Mon_1 on π_1/Γ is trivial.

Proof. As Mon_1 is an automorphism of π_1 , it preserves L_2 . Also, $Mon_1(\gamma) = \gamma$, $Mon_1(\delta) = \gamma\delta_0\gamma\delta_1\gamma\delta_2\gamma\delta_3 = \delta \bmod \Gamma \in \Gamma$, so Mon_1 preserves Γ . Also $Mon_1(\delta_i) = \gamma\delta_i = \delta_i \bmod \Gamma$, which proves the last statement. \square

As $\Gamma \cap L_2 \subset K$, this implies that Mon_1 acts trivially on L_2/K and therefore can be disregarded.

Corollary 2.3. \mathcal{O} is generated by γ and $Mon_0^i(\delta)$, $i = 0, 1, \dots$

Lemma 2.4. Mon_0 preserves $L_i \cap \langle \delta_0, \delta_1, \delta_2, \delta_3 \rangle$ and the induced action of Mon_0 on $L_i \cap \langle \delta_0, \delta_1, \delta_2, \delta_3 \rangle / (L_{i+1} \cap \langle \delta_0, \delta_1, \delta_2, \delta_3 \rangle)$ is trivial.

Proof. Follows immediately from Lemma 2.1. \square

So we have to investigate the orbit of δ in the free group generated by $\langle \delta_i, i = 0, 1, 2, 3 \rangle$, under the action of Mon_0 given by Lemma 2.1.

Define $M(\sigma) = \delta_1^{-1}\delta_0^{-1}Mon_0(\sigma)\delta_0\delta_1$ on $\langle \delta, \delta_1, \delta_2, \delta_3 \rangle$,

$$M = \{\delta_0 \mapsto \delta_1^{-1}\delta_0\delta_1, \delta_1 \mapsto \delta_1, \delta_2 \mapsto \delta_2, \delta_3 \mapsto [\delta_2, \delta_3]\delta_3\}, \quad (2.3)$$

and define $Var(\sigma) = M(\sigma)\sigma^{-1}$. Note that for $\sigma \in \mathcal{O}$

$$Var(\sigma) = [\delta_1^{-1}\delta_0^{-1}, Mon_0(\sigma)]Mon_0(\sigma)\sigma^{-1} = Mon_0(\sigma)\sigma^{-1} \bmod K \in \mathcal{O}. \quad (2.4)$$

This means that both Mon_0 and M generate the same orbit, and one can use either of them. However, M is computationally more convenient. We formally define $Var(\gamma) = \delta$ and $Var^{i+1}(\gamma) = Var^i(\delta)$.

Corollary 2.5. *For any $i \geq 1$, $Var^i(\gamma) \in L_i \cap \langle \delta, \delta_1, \delta_2, \delta_3 \rangle$.*

Proof. We have $Var(\gamma) = \delta \in \pi_1 = L_1$. Therefore $Var^i(\gamma)$ belong to the subgroup generated by δ_i . By induction, and using Lemma 2.4 and (2.4), we see that $Mon_0(Var^i(\gamma)) = Var^i(\gamma) \bmod L_{i+1} \cap \langle \delta, \delta_1, \delta_2, \delta_3 \rangle$. Therefore $Var^{i+1}(\gamma) \in L_{i+1} \cap \langle \delta, \delta_1, \delta_2, \delta_3 \rangle$. \square

Lemma 2.6. *Any element $w \in \mathcal{O}$ can be represented as*

$$w = \gamma^{n_0} Var(\gamma)^{n_1} \dots Var^i(\gamma)^{n_k} \bmod K, \quad i = i(w). \quad (2.5)$$

Proof. Indeed, any element in \mathcal{O}/K is a product of $M^i(\gamma)$, and these elements can be represented in this form: if

$$M^i(\gamma) = \gamma^{n_0} \delta^{n_1} \dots Var^i(\gamma)^{n_k} \bmod K, \quad (2.6)$$

then

$$\begin{aligned} M^{i+1}(\gamma) &= M(\gamma^{n_0})M(\delta^{n_1}) \dots M(Var^i(\gamma)^{n_k}) \bmod K \\ &= (Var(\gamma)\gamma)^{n_0} (Var(\delta)\delta)^{n_1} \dots (Var^{i+1}(\gamma)Var^i(\gamma))^{n_k} \bmod K \\ &= \gamma^{n_0} \delta^{n_0+n_1} \dots Var^i(\gamma)^{n_{k-1}+n_k} Var^{i+1}(\gamma)^{n_k} \bmod K, \end{aligned}$$

as $Var^i(\gamma)$ commute modulo K . \square

Define by induction the maps $d_k : \pi_1 \rightarrow \pi_1$ as $d_1 = Id$, and $d_{k+1}(\sigma) = [\delta_2, d_k(\sigma)]$. Note that $d_k(\sigma) = [\delta_2, [\delta_2, [\dots [\delta_2, \sigma] \dots]] \in L_k$ for all $k \geq 1$, $\sigma \in \pi_1$.

Proposition 2.7. *Denote $x = \delta_1\delta_2$, $z = \delta_2\delta_3$ and define*

$$v_1 = \delta, \quad v_k = [x, d_{k-1}(z)] \text{ for } k \geq 2. \quad (2.7)$$

Then $v_i \in \mathcal{O}$ and $Var^i(\gamma) = v_i \bmod K$.

Before proving Proposition 2.7, we prove

Lemma 2.8. $d_{k-1}([\delta_2, z]z) = d_k(z)d_{k-1}(z)$.

Proof. By induction,

$$\begin{aligned} d_k([\delta_2, z]z) &= [\delta_2, d_{k-1}([\delta_2, z]z)] = [\delta_2, d_k(z)d_{k-1}(z)] = \\ &= [\delta_2, d_k(z)][\delta_2, d_{k-1}(z)] [[d_{k-1}(z), \delta_2], d_k(z)] = \\ &= d_{k+1}(z)d_k(z)[d_k(z)^{-1}, d_k(z)] = d_{k+1}(z)d_k(z). \quad \square \end{aligned}$$

Proof of Proposition 2.7. From Lemma 2.1 we see that $v_1 = Mon_0(\gamma)\gamma^{-1} = \delta$.

Note that by (2.3)

$$M(x) = x, \quad M(\delta_2) = \delta_2, \quad M(z) = \delta_2 z \delta_2^{-1} = [\delta_2, z]z. \quad (2.8)$$

We have

$$\delta = \delta_1 \delta_2 \delta_3 \delta_0 [\delta_0^{-1}, \delta^{-1}] = \delta_1 \delta_2 \delta_3 \delta_0 \bmod K, \quad (2.9)$$

so, modulo K ,

$$\begin{aligned} Var^2(\gamma) &= M(\delta_1 \delta_2 \delta_3 \delta_0) \delta^{-1} = \delta_1 \cdot \delta_2 \cdot \delta_2 \delta_3 \delta_2^{-1} \cdot \delta_1^{-1} \delta_0 \delta_1 \cdot \delta^{-1} = \\ &= [\delta_1 \delta_2, \delta_2 \delta_3] [\delta_2 \delta_3, \delta] = [x, z] = v_2. \end{aligned} \quad (2.10)$$

In particular, $v_2 \in \mathcal{O}$.

For the third variation, again modulo K ,

$$\begin{aligned} \text{Var}^3(\gamma) &= M([x, z])v_2^{-1} = [x, [\delta_2, z]z]v_2^{-1} = \\ &= [x, [\delta_2, z]][x, z][[z, x], [\delta_2, z]]v_2^{-1} = [x, [\delta_2, z]]. \end{aligned} \quad (2.11)$$

as $[[z, x], [\delta_2, z]] = [v_2^{-1}, [\delta_2, z]] \in K$.

Now, from (2.8) follows $M(v_k) = [x, d_{k-1}([\delta_2, z]z)]$, so modulo K ,

$$\begin{aligned} \text{Var}^{k+1}(\gamma) &= M(v_k)v_k^{-1} = [x, d_k(z)d_{k-1}(z)]v_k^{-1} = \\ &= [x, d_k(z)][x, d_{k-1}(z)] [[d_{k-1}(z), x], d_k(z)]v_k^{-1} = \\ &= [x, d_k(z)]v_k[v_k^{-1}, d_k(z)]v_k^{-1} = [x, d_k(z)] = v_{k+1}. \quad \square \end{aligned}$$

Proposition 2.9. $\mathcal{O} = \langle \gamma, v_i, i = 1, \dots \rangle$.

Proof. Denote $\mathcal{O}^v = \langle \gamma, v_i, i = 1, \dots \rangle$ the normal subgroup of π_1 generated by γ, v_i . It follows from Proposition 2.7 that $v_i \in \mathcal{O}$, so $\mathcal{O}^v \subset \mathcal{O}$.

Let us prove the opposite inclusion. By Lemma 2.6, $\mathcal{O} = \langle \gamma, \text{Var}^i(\gamma), i = 1, \dots \rangle$. By Lemma 2.7,

$$\text{Var}^i(\gamma) = v_i W_{i,1}(\{[a_j^1, \text{Var}^j(\gamma)]\}_{j \geq 0}), \quad i \geq 0, \quad (2.12)$$

where $W_{i,1}$ are some words. Substituting these equalities into their right hand sides, we get

$$\text{Var}^i(\gamma) = v_i W_{i,2}(\{[b_j, v_j], [a_j^2, [a_j^1, \text{Var}^j(\gamma)]]\}_{j \geq 0}), \quad i \geq 0. \quad (2.13)$$

Repeating substitution $\ell - 1$ times, we get for any $\ell \geq 1$

$$\text{Var}^i(\gamma) = v_i \epsilon_{i,\ell} w_{i,\ell}, \quad \epsilon_{i,\ell} \in \mathcal{O}^v, \quad w_{i,\ell} \in [\pi_1, [\pi_1, [\dots [\pi_1, \mathcal{O}] \dots]] \subset L_{\ell+1}. \quad (2.14)$$

This implies that $\mathcal{O}L_{\ell+1} \subset \mathcal{O}^v L_{\ell+1}$ for any $\ell \geq 1$. This implies that $\mathcal{O} \subset \mathcal{O}^v$. \square

Corollary 2.10. $K = \langle [\pi_1, \gamma], [\pi_1, v_i], i = 1, \dots \rangle$.

2.2. Depth is infinite. Here we prove Theorem 1.5. The main idea is to construct for any k a matrix representation ρ_k of π_1 sending all generators of \mathcal{O} except v_{k+2} to identity. We prove that $\rho_k(v_{k+2}) \notin \rho_k(K)$ for a generic choice of parameters a, c of the representation ρ_k , which implies Theorem 1.5.

Let $A_0 = a$, $B_0 = 1$ and $C_0 = c$, where $a, c \neq 0, 1$. Define inductively the $2^k \times 2^k$ -matrices A_k, B_k, C_k as follows:

$$A_{k+1} = \begin{bmatrix} A_k & 0 \\ 0 & \mathbb{I} \end{bmatrix}, B_{k+1} = \begin{bmatrix} B_k & \mathbb{I} \\ 0 & B_k \end{bmatrix}, C_{k+1} = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & C_k \end{bmatrix}, \quad (2.15)$$

where \mathbb{I} is the corresponding identity matrix.

Proposition 2.11. *Let $x = \delta_1 \delta_2$ and $z = \delta_2 \delta_3$ be as in Proposition 2.7. Consider the representation $\rho_k : \pi_1 \rightarrow GL(2^k)$ defined by $\rho_k(\gamma) = \rho_k(\delta) = \mathbb{I}$, $\rho_k(x) = A_k$, $\rho_k(\delta_2) = B_k$ and $\rho_k(z) = C_k$. Then*

- (1) $\rho_k(v_i) = \mathbb{I}$ for $i \neq k+2$ and $\rho_k(v_{k+2}) \neq \mathbb{I}$, and
- (2) $\rho_k(v_{k+2}) \notin [\rho_k(v_{k+2}), \rho_k(\pi_1)] = \rho_k(K)$ for generic a, c .

Remark 2.12. As π_1 is a free group and $\gamma, \delta, x, \delta_2, z$ are its generators, such a representation ρ_k exists.

Proof. We start with another description of A_k, B_k and C_k . Let

$$\mathbb{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Recall that tensor products are multiplied factorwise:

$$(X_1 \otimes Y_1)(X_2 \otimes Y_2) = X_1 X_2 \otimes Y_1 Y_2.$$

Denote by $b_{j_1 \dots j_l}$ to be the tensor product of $(k-l)$ copies of \mathbb{I}_2 and l copies of J_2 , with J_2 being exactly the $j_1^{\text{th}}, \dots, j_l^{\text{th}}$ factors. Similarly, denote by $e_{j_1 \dots j_l}$ to be the tensor product of $(k-l)$ copies of E_2 and l copies of J_2 , with J_2 being exactly the $j_1^{\text{th}}, \dots, j_l^{\text{th}}$ factors. Finally, denote $\alpha = F_2^{\otimes k}$, $\gamma = E_2^{\otimes k}$ and $\beta = \sum_{j=1}^k b_j$.

Using this notations, we have

$$A_k = \mathbb{I}_{2^k} + (a-1)\alpha, \quad B_k = \mathbb{I}_{2^k} + \beta \quad \text{and} \quad C_k = \mathbb{I}_{2^k} + (c-1)\gamma. \quad (2.16)$$

Our immediate goal is to compute $[B_k, C_k]$. Evidently,

$$A_k^{-1} = \mathbb{I}_{2^k} + \left(\frac{1}{a} - 1\right)\alpha, \quad C_k^{-1} = \mathbb{I}_{2^k} + \left(\frac{1}{c} - 1\right)\gamma. \quad (2.17)$$

As $J_2^2 = 0$, we have $b_{j_1 \dots j_l} b_{j'_1 \dots j'_l}$ equals $b_{j_1 \dots j_l j'_1 \dots j'_l}$ if the sets $\{j_1 \dots j_l\}, \{j'_1 \dots j'_l\}$ do not intersect, and zero otherwise.

Therefore

$$\beta^l = l! \sum_{1 \leq j_1 < \dots < j_l \leq k} b_{j_1 \dots j_l}, \quad \beta^k = k! J_2^{\otimes k} \quad \text{and} \quad \beta^{k+1} = 0. \quad (2.18)$$

In particular,

$$B_k^{-1} = \mathbb{I}_{2^k} + \tilde{\beta}, \quad \text{where} \quad \tilde{\beta} = -\beta + \beta^2 - \dots \pm \beta^k.$$

Now, from $N_2^2 = N_2$, $N_2 J_2 = 0$, $J_2 N_2 = J_2$ we have

$$\gamma^2 = \gamma, \quad \gamma \beta^l = \gamma \tilde{\beta} = 0, \quad (2.19)$$

and

$$\beta^l \gamma = \epsilon^{[l]} = l! \sum_{1 \leq j_1 < \dots < j_l \leq k} e_{j_1 \dots j_l}, \quad (2.20)$$

Note that $\epsilon^{[l]}\epsilon^{[l']} = 0$ for all $l, l' \geq 1$, $\epsilon^{[k]} = k! J_2^{\otimes k}$ and $\epsilon^{[l]} = 0$ for $l > k$.

Now, using the above formulae we see that

$$\begin{aligned} [B_k, C_k] &= (\mathbb{I}_{2^k} + \beta)(\mathbb{I}_{2^k} + (c-1)\gamma)(\mathbb{I}_{2^k} + \tilde{\beta})(\mathbb{I}_{2^k} + (\frac{1}{c} - 1)\gamma) \\ &= \mathbb{I}_{2^k} - (\frac{1}{c} - 1)\beta\gamma = \mathbb{I}_{2^k} - (\frac{1}{c} - 1)\epsilon^{[1]}, \end{aligned} \quad (2.21)$$

and, as $(\epsilon^{[1]})^2 = 0$,

$$[B_k, C_k]^{-1} = \mathbb{I}_{2^k} + (\frac{1}{c} - 1)\epsilon^{[1]}. \quad (2.22)$$

Continuing,

$$[B_k, [B_k, C_k]] = \mathbb{I}_{2^k} - (\frac{1}{c} - 1)\beta\epsilon^{[1]} = \mathbb{I}_{2^k} - (\frac{1}{c} - 1)\epsilon^{[2]}, \quad (2.23)$$

and, by induction, for a commutator with l entries of B_k ,

$$[B_k, [\dots [B_k, C_k] \dots]] = \mathbb{I}_{2^k} - (\frac{1}{c} - 1)\epsilon^{[l]}. \quad (2.24)$$

Now, similarly, from

$$F_2^2 = F_2, \quad F_2 J_2 = J_2, \quad J_2 F_2 = 0 \quad (2.25)$$

we have

$$\begin{aligned} \alpha^2 &= \alpha, & \epsilon^{[l]}\alpha &= 0 \text{ for all } l, \\ \alpha\epsilon^{[k]} &= \epsilon^{[k]} = k! J_2^{\otimes k}, & \alpha\epsilon^{[l]} &= 0 \text{ for } l \neq k. \end{aligned}$$

Therefore

$$\rho_k(v_{l+2}) = [A_k, [B_k, [\dots [B_k, C_k] \dots]] = \mathbb{I}_{2^k} + (\frac{1}{c} - 1)(\frac{1}{a} - 1)\alpha\epsilon^{[l]}, \quad (2.26)$$

i.e.

$$\begin{aligned} \rho_k(v_{l+2}) &= \mathbb{I}_{2^k} & \text{for } l+2 &\neq k, \\ \rho_k(v_{k+2}) &= \mathbb{I}_{2^k} + (\frac{1}{c} - 1)(\frac{1}{a} - 1)k! J_2^{\otimes k} & \neq \mathbb{I}_{2^k}, \end{aligned} \quad (2.27)$$

which proves the first claim of Proposition 2.11.

Let $s = \prod g_i^{m_i} \in \pi_1$, where $g_i \in \{\gamma, \delta, x, \delta_2, z\}$ and $m_i \in \mathbb{Z}$. Then $\rho_k(s) = D + U$, where U is a strictly upper triangular matrix and $D = \text{diag}(a^m, 1, \dots, 1, c^n)$. Therefore

$$[\rho_k(s), \rho_k(v_{k+2})] = \mathbb{I}_{2^k} + \left(\frac{a^m}{c^n} - 1\right) \left(\frac{1}{c} - 1\right) \left(\frac{1}{a} - 1\right) k! J_2^{\otimes k}. \quad (2.28)$$

Now, assume

$$\rho_k(v_{k+2}) = \prod_j [\rho_k(s_j), \rho_k(v_{k+2})]. \quad (2.29)$$

By (2.27), (2.28), we have

$$\mathbb{I}_{2^k} + \left(\frac{1}{c} - 1\right) \left(\frac{1}{a} - 1\right) k! J_2^{\otimes k} = \mathbb{I}_{2^k} + \left(\sum \left(\frac{a^{m_j}}{c^{n_j}} - 1\right)\right) \left(\frac{1}{c} - 1\right) \left(\frac{1}{a} - 1\right) k! J_2^{\otimes k}, \quad (2.30)$$

or, equivalently, $1 = \sum \left(\frac{a^{m_j}}{c^{n_j}} - 1\right)$. Collecting similar terms, we get

$$\sum \lambda_i \frac{a^{m_i}}{c^{n_i}} = 1 + \sum \lambda_i, \quad \text{where } \lambda_i, m_i, n_i \in \mathbb{Z},$$

and for any i one of the exponents m_i, n_i is non-zero. This cannot hold for all a, c : if the left hand side is a constant, then all λ_i vanish, and we get $0 = 1$. Therefore any representation (2.29) fails on a Zariski open subset of $\mathbb{C}_{(a,c)}$, so all such representations fail for a generic choice of a, c . \square

Proof of Theorem 1.5. By Corollary 2.10 and Proposition 2.11(1), we have $\rho_k(K) = [\rho_k(\pi_1), \rho_k(v_{k+2})]$. By Proposition 2.11(2), $\rho_k(v_{k+2}) \notin [\rho_k(\pi_1), \rho_k(v_{k+2})]$ for a generic choice of a, c . This means that $(\mathcal{O} \cap L_{k+2}) \setminus K$ contains v_{k+2} , so is non-empty for all k . \square

3. FIRST NONZERO MELNIKOV FUNCTION OF LENGTH 3

3.1. Cohomologies: notations. Denote $f_1 = x + 1$, $f_2 = y - 1$, $f_3 = x - 1$ and $f_4 = y + 1$. Denote $\phi_i = \log f_i$ and $\eta_i = d\phi_i = \frac{df_i}{f_i}$.

The cycles $\gamma, \delta, \delta_1, \delta_2, \delta_3$ form a basis of $H_1(\Gamma_t)$, and γ, δ form a basis of the orbit of γ in $H^1(\Gamma_t)$. As ϕ_i are univalued on γ , the restrictions to Γ_t of polynomial forms $\{F\eta_i\}_{i=1}^3$ lie in the orthogonal complement $\mathcal{O}^\perp \subset H^1(\Gamma_t)$ of the orbit $\mathcal{O} \subset H_1(\Gamma_t)$ of γ in $H_1(\Gamma_t)$, and in fact form its basis. We have

$$\begin{aligned} \int_{\delta_1} \eta_1 &= 0, & \int_{\delta_1} \eta_2 &= 0, & \int_{\delta_1} \eta_3 &= 2\pi i \\ \int_{\delta_2} \eta_1 &= 0, & \int_{\delta_2} \eta_2 &= 2\pi i, & \int_{\delta_2} \eta_3 &= -2\pi i \\ \int_{\delta_3} \eta_1 &= 2\pi i, & \int_{\delta_3} \eta_2 &= -2\pi i, & \int_{\delta_3} \eta_3 &= 0. \end{aligned} \quad (3.1)$$

Note that $d\eta_i = 0$, so the Gelfand-Leray derivatives $\frac{d\eta_i}{dF}$ vanish.

3.2. Linear perturbations. Consider the rational 1-form of type (1.7), i.e.

$$\omega = a_1(F)\eta_1 + a_2(F)\eta_2 + a_3(F)\eta_3, \quad (3.2)$$

where $a_i(t)$ are holomorphic on τ , and consider the perturbation

$$dF + \epsilon\omega = 0, \quad F = (x^2 - 1)(y^2 - 1). \quad (3.3)$$

Remark 3.1. Note that restriction to Γ_t of any form ω such that $\int_{\gamma(t)} \omega \equiv 0$ is cohomologous to a linear combination of η_i .

The Poincaré map along the cycles γ is

$$P_\gamma(t) = t + \epsilon M_{\gamma,1}(t) + \epsilon^2 M_{\gamma,2}(t) + \epsilon^3 M_{\gamma,3}(t) + \dots, \quad (3.4)$$

with $M_{\gamma,1}(t) = \int_{\gamma(t)} \omega \equiv 0$. Our goal is to find a polynomial form ω providing the highest possible order of the first non-vanishing Melnikov function $M_{\gamma,i}(t)$

Proposition 3.2. $M_{\gamma,2} \equiv 0$ and $M_{v_3,3} \neq 0$ if, and only if,

$$\begin{aligned} a_3(t) &= \alpha_1 \int_0^t \frac{\alpha_2(\tau)}{\alpha_1^2(\tau)} d\tau + c_0 \alpha_1 \\ a_1(t) &= a_3(t) + \alpha_1(t) \\ a_2(t) &= \lambda \alpha_1, \end{aligned} \tag{3.5}$$

where $\lambda \in \mathbb{C}^*$ and $\alpha_1(t)$ and $\alpha_2(t)$ are linearly independent functions over \mathbb{C} , and α_1 is not constant.

To prove Proposition 3.2, we consider the second and third variations of γ , i.e. v_2 and v_3 from Proposition 2.7, and the corresponding Poincaré maps P_{v_2} and P_{v_3} . Proposition 3.4 implies that

$$\begin{aligned} P_{v_2}(t) &= t + \epsilon^2 M_{v_2,2}(t) + O(\epsilon^3), \\ P_{v_3}(t) &= t + \epsilon^3 M_{v_3,3}(t) + O(\epsilon^4), \end{aligned} \tag{3.6}$$

and provides explicit expression of $M_{v_2,2}(t)$, $M_{v_3,3}(t)$ in terms of coefficients $a_i(F)$. This allows to find conditions on a_i guaranteeing $M_{v_3,3}(t) \neq 0$ and $M_{v_2,2}(t) \equiv 0$. We prove that the last condition is equivalent to $M_{\gamma,2}(t) \equiv 0$ in Lemma 3.7.

Remark 3.3 (Geometric interpretation of Proposition 3.2). The forms (3.2) form a three dimensional module Ω over the ring of germs of holomorphic functions at t . The Poincaré map along γ is a map $\mathcal{P}_\gamma : U \subset \Omega \rightarrow \mathcal{H}ol(\tau)$, where $\mathcal{H}ol(\tau)$ is the set of germs of holomorphic mappings $g : (\tau, p_0) \rightarrow \tau$.

The perturbations (1.1) are germs of lines in Ω , and the order of the first non-zero Melnikov function of the perturbation can be interpreted as the order of vanishing of \mathcal{P} on these lines, i.e. the order of tangency of these lines to the set $\{\mathcal{R} = 0\}$ of integrable perturbations. Theorem 1.6 claims that the maximum order of this tangency is either at most three or the line lies entirely in $\{\mathcal{R} = 0\}$.

To construct the perturbations with first non-zero Melnikov function M_k of higher length, we necessarily have to increase k , i.e. the order of tangency of the perturbation with the set of integrable foliations. This means that we have either to consider non-linear perturbations, i.e. germs of curves in Ω , or consider a wider class $\tilde{\Omega}$ of perturbations, e.g. by including relatively exact forms.

Still, the first non-zero Melnikov function of a non-linear perturbation

$$dF + \epsilon \omega_1 + \epsilon^2 \omega_2 + \dots = 0, \quad \omega_i \in \Omega \tag{3.7}$$

can be of high order, but of small length. It is easy to see that the terms of highest length of the corresponding Melnikov functions depend only on ω_1 . Thus, to ensure that the length of the first non-vanishing Melnikov functions is at least 4, we should take ω_1 such that $dF + \epsilon \omega_1 = 0$ is integrable (otherwise $M_3 \neq 0$), and find non-linear terms in such a way that $M_{\gamma,4} \equiv 0$ (as its longest terms are determined by ω_1 , they necessarily vanish, so its length could be at most 3), but $M_{\gamma,5} \neq 0$ and has length 4 (it cannot be of length 5 by the same reason). The latter would follow from $M_{v_4,5} \neq 0$. This program can be realized, but it is computationally hard. Moreover, it is not clear how one can generalize this approach to higher length, so we omit the computations.

3.3. Poincaré maps as time-one flows of vector fields on the transversal.

Consider the family (1.1) as a one-dimensional foliation

$$\mathcal{F} = \{dF + \epsilon \omega = 0, d\epsilon = 0\} \tag{3.8}$$

in $\mathbb{C}_{x,y,\epsilon}^3$. Let $\Theta = \tau \times (\mathbb{C}_\epsilon, 0)$ be a transversal to the algebraic leaf $\Gamma = \Gamma_t \times \{0\}$ at the point $(p_0, 0)$, and denote $\mathcal{D} = \text{Diff}((\Theta, (p_0, 0)))$ the group of germs of holomorphic diffeomorphisms of Θ . Holonomy of \mathcal{F} along various paths $\gamma \in \pi_1(\Gamma, (p_0, 0))$ defines

a representation $\tilde{P} : \pi_1 \rightarrow \mathcal{D}$ preserving ϵ , i.e. $\tilde{P}_\gamma : (x, \epsilon) \rightarrow (P_\gamma(x, \epsilon), \epsilon)$ for any $\gamma \in \pi_1$.

Define $v_\gamma = (d\tilde{P}_\gamma)(\partial_\epsilon)$, $v_e = \partial_\epsilon$, and let ϕ_γ^s be the s -time flow of v_γ (necessarily $L_{v_\gamma}(\epsilon) = 1$). By definition, \tilde{P}_γ conjugates flows of v_e and v_γ . In particular, for all $p \in \tau$

$$\tilde{P}_\gamma(p, \epsilon) = \tilde{P}_\gamma(\phi_\epsilon^\epsilon(p, 0)) = \phi_\gamma^\epsilon(\tilde{P}_\gamma(p, 0)) = \phi_\gamma^\epsilon(p, 0), \quad (3.9)$$

as $\tilde{P}_\gamma(p, 0) \equiv (p, 0)$ for all $\gamma \in \pi_1$.

Let $(t = F(p), \epsilon)$ be parameterization of Θ . The expansion (1.2) is the expansion of \tilde{P}_γ in degrees of ϵ ,

$$\tilde{P}_\gamma(t, \epsilon) = (t + \epsilon^\mu M_{\gamma, \mu}(t) + o(\epsilon^\mu), \epsilon). \quad (3.10)$$

Let

$$v_\gamma = (v_\gamma^0(t) + \epsilon v_\gamma^1(t) + \dots) \partial_t + \partial_\epsilon \quad (3.11)$$

be decomposition of v_γ . Evidently, $v_\gamma^0 = \dots = v_\gamma^{\mu-2} \equiv 0$, and $v_\gamma^{\mu-1}(t) = M_{\gamma, \mu}(t)$.

Proposition 3.4. *Let $\gamma_1, \gamma_2, \gamma = [\gamma_1, \gamma_2] \in \pi_1$, and let*

$$P_{\gamma_i}(t, \epsilon) = t + \epsilon^{\mu_i} M_{\gamma_i, \mu_i} + o(\epsilon^{\mu_i}), \quad i = 1, 2,$$

with $M_{\gamma_1, \mu_1}, M_{\gamma_2, \mu_2} \not\equiv 0$.

Then $v_\gamma^0 = \dots = v_\gamma^{\mu-2} \equiv 0$ for $\mu = \mu_1 + \mu_2$, and

$$v_{[\gamma_1, \gamma_2]}^{\mu-1}(t) = W(M_{\gamma_1, \mu_1}(t), M_{\gamma_2, \mu_2}(t)), \quad (3.12)$$

where $W(f, g) = fg' - f'g$ denotes the Wronskian of f, g .

Alternatively,

$$v_{[\gamma_1, \gamma_2]}^{\mu-1}(t) \partial_t = [M_{\gamma_1, \mu_1}(t) \partial_t, M_{\gamma_2, \mu_2}(t) \partial_t], \quad (3.13)$$

where brackets denote the Lie bracket of vector fields.

Remark 3.5. Essentially, (1.1) induces a homomorphism \mathfrak{R} of the fundamental group π_1 of Γ_t to the group of germs at identity of analytic curves in the groupoid $Diff(\tau)$.

More precisely, for any $\gamma \in \pi_1$ we get a germ $\mathfrak{R}(\gamma)$ at identity of an analytic curve

$$\{\hat{\omega}_\gamma\} = \left\{ \tilde{P}_\gamma(\cdot, \epsilon) \right\} \subset Diff(\tau).$$

The Lie algebra \mathcal{X} of $Diff(\tau)$ "is" the Lie algebra of germs at p_0 of vector fields on τ , and v_γ defines the corresponding (under exponential map) path $\log \hat{\omega}_\gamma$ in this Lie algebra. The path $\tilde{P}_\gamma(\cdot, \epsilon)$ is not necessarily a one-parametric group, and the path $\log \hat{\omega}_\gamma$ is not necessarily constant, but we are interested in the leading term of $\hat{\omega}_\gamma$ only. If $\mu = 1$, then the leading term is the tangent vector $M_{\gamma, 1} \partial_t$ to $\hat{\omega}_\gamma$. However, if $\mu > 1$ then the tangent vector is zero.

To include the case $\mu > 1$ consider the group \mathfrak{G} of germs at identity of analytic curves in the groupoid $Diff(\tau)$. \mathfrak{G} has natural filtration by order of tangency of the germ to the constant germ, i.e. by the order of the first non-zero term in its Taylor decomposition in ϵ , which induces a filtration on its Lie algebra, and the associated graded algebra is a Lie algebra $\hat{\mathfrak{G}}$ isomorphic to $\mathcal{X} \otimes \mathbb{C}[[\epsilon]]$, up to a shift of grading by 1. The homomorphism \mathfrak{R} pulls back the above filtration of \mathfrak{G} to a filtration of π_1 , compatible with the group commutator (for generic perturbations this filtration most probably coincides with the lower central series L_i). Starting from this filtration on π_1 , one can build a Lie algebra in a standard way, and Proposition 3.4 shows that \mathfrak{R} lifts to a Lie algebra mapping between this Lie algebra and $\hat{\mathfrak{G}}$.

Proof. The monodromy of $\gamma = \gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1}$ is given by $P_\gamma = P_{\gamma_2}^{-1} \circ P_{\gamma_1}^{-1} \circ P_{\gamma_2} \circ P_{\gamma_1}$. Denote $P_{\gamma_i}(t, \epsilon) = t + \epsilon^{\mu_i} M_{\gamma_i, \mu_i} + \dots + \epsilon^\mu M_{\gamma_i, \mu} + o(\epsilon^\mu)$ for $i = 1, 2$. Then

$$\begin{aligned} P_{\gamma_1} \circ P_{\gamma_2}(t, \epsilon) &= P_{\gamma_2}(t, \epsilon) + \sum_{j=\mu_1}^{\mu} \epsilon^j M_{\gamma_1, j}(P_{\gamma_2}(t, \epsilon)) + o(\epsilon^\mu) = \\ &= t + \sum_{j=\mu_2}^{\mu} \epsilon^j M_{\gamma_2, j}(t) + \sum_{j=\mu_1}^{\mu} \epsilon^j M_{\gamma_1, j}(t) + \epsilon^\mu M'_{\gamma_1, \mu_1} M_{\gamma_2, \mu_2} + o(\epsilon^\mu). \end{aligned}$$

Similarly,

$$P_{\gamma_2} \circ P_{\gamma_1}(t, \epsilon) = t + \sum_{j=\mu_1}^{\mu} \epsilon^j M_{\gamma_1, j}(t) + \sum_{j=\mu_2}^{\mu} \epsilon^j M_{\gamma_2, j}(t) + \epsilon^\mu M'_{\gamma_2, \mu_2} M_{\gamma_1, \mu_1} + o(\epsilon^\mu),$$

and therefore

$$P_{\gamma_2} \circ P_{\gamma_1}(t, \epsilon) = P_{\gamma_1} \circ P_{\gamma_2}(t, \epsilon) + \epsilon^\mu (M_{\gamma_1, \mu_1} M'_{\gamma_2, \mu_2} - M'_{\gamma_1, \mu_1} M_{\gamma_2, \mu_2}) + o(\epsilon^\mu).$$

As $(P_{\gamma_2}^{-1} \circ P_{\gamma_1}^{-1})' = 1 + O(\epsilon)$, application of $(P_{\gamma_2}^{-1} \circ P_{\gamma_1}^{-1}(s, \epsilon), \epsilon)$ provides the required equality

$$P_\gamma = P_{\gamma_2}^{-1} \circ P_{\gamma_1}^{-1} \circ P_{\gamma_2} \circ P_{\gamma_1}(t, \epsilon) = t + \epsilon^\mu (M_{\gamma_1, \mu_1} M'_{\gamma_2, \mu_2} - M'_{\gamma_1, \mu_1} M_{\gamma_2, \mu_2}) + o(\epsilon^\mu). \quad \square$$

3.4. Explicit computations. By Poincaré-Pontryagin criterion, $M_{\sigma, 1} = \int_\sigma \omega$. From (3.1) we have

$$\begin{aligned} (2\pi i)^{-1} \int_{\delta_1 + \delta_2} \omega &= a_2(t), \\ (2\pi i)^{-1} \int_{\delta_2(t)} \omega &= a_2(t) - a_3(t), \\ (2\pi i)^{-1} \int_{\delta_2 + \delta_3} \omega &= a_1(t) - a_3(t). \end{aligned}$$

By Proposition 3.4 we have

$$\begin{aligned} (2\pi i)^{-2} M_{v_2, 2}(t) &= W(a_2(t), a_1(t) - a_3(t)), \\ (2\pi i)^{-3} M_{v_3, 3}(t) &= W(a_2(t), W(a_2(t) - a_3(t), a_1(t) - a_3(t))). \end{aligned} \quad (3.14)$$

In what follows, the $2\pi i$ factors are not important, so we will omit them.

Lemma 3.6. $M_{v_2, 2} \equiv 0$ and $M_{v_3, 3} \neq 0$ if, and only if,

$$\begin{aligned} a_3(t) &= \alpha_1 \int_0^t \frac{\alpha_2(\tau)}{\alpha_1^2(\tau)} d\tau + c_0 \alpha_1 \\ a_1(t) &= a_3(t) + \alpha_1(t) \\ a_2(t) &= \lambda \alpha_1(t), \text{ with } \lambda \in \mathbb{C}^*, \end{aligned} \quad (3.15)$$

where $\alpha_1(t)$ and $\alpha_2(t)$ are linearly independent functions over \mathbb{C} , and α_1 is not constant.

Proof. From $M_{v_3, 3} \neq 0$ we see that $a_2(t), a_1(t) - a_3(t) \neq 0$.

Then $M_{v_2, 2} \equiv 0$ is equivalent to

$$a_2(t) = \lambda_1 (a_1(t) - a_3(t)), \text{ for some } \lambda_1 \in \mathbb{C}^*. \quad (3.16)$$

This implies, by linearity of Wronskians,

$$M_{v_3, 3} = W(a_2(t), W(a_3(t), a_1(t))) = \lambda_1 W(a_1(t) - a_3(t), W(a_3(t), a_1(t))). \quad (3.17)$$

Then, $M_{v_3, 3} \neq 0$ if, and only if,

$$W(a_1, a_3) \neq \lambda_2 (a_1(t) - a_3(t)), \text{ for all } \lambda_2 \in \mathbb{C}. \quad (3.18)$$

In other words,

$$\begin{aligned} a_1(t) - a_3(t) &= \alpha_1(t) \\ a'_3(t)a_1(t) - a'_1(t)a_3(t) &= \alpha_2(t), \end{aligned} \quad (3.19)$$

where $\alpha_1(t)$ and $\alpha_2(t)$ are linearly independent functions over \mathbb{C} , and α_1 is not constant, in order to get condition (3.18).

The solution of the system (3.19) is

$$\begin{aligned} a_1(t) &= a_3(t) + \alpha_1(t) \\ a_3(t) &= \alpha_1 \int_0^t \frac{\alpha_2(\tau)}{\alpha_1^2(\tau)} d\tau + c_0 \alpha_1, \quad c_0 \in \mathbb{C}. \end{aligned}$$

Substituting a_1 and a_3 in expression (3.16) we get $a_2(t)$. \square

In general, $M_{v_2,2} \equiv 0$ does not necessarily imply $M_{\gamma,2}(t) \equiv 0$. However,

Lemma 3.7. *For a form ω of form (3.2), the condition $M_{v_2,2}(t) \equiv 0$ is equivalent to $M_{\gamma,2}(t) \equiv 0$.*

Proof. Evidently, if $M_{\gamma,2}(t) \equiv 0$ then $M_{v_2,2}(t) = \text{Var}(M_{\gamma,2}(t)) \equiv 0$, so one implication is trivial.

Since $M_{\gamma,1} \equiv 0$ by Françoise algorithm we have that $M_{\gamma,2} = \int_{\gamma} \omega' \omega$. Using integration by parts, we can rewrite ω as

$$\omega = \sum_{i=1}^3 a_i(t) d\phi_i = - \sum_{i=1}^3 \phi_i a'_i(F) dF + d \left(\sum_{i=1}^3 a_i(F) \phi_i \right), \quad \phi_i = \log f_i.$$

Denote $g = - \sum_{i=1}^3 \phi_i a'_i(F)$ and $R = \sum_{i=1}^3 a_i(F) \phi_i$. Then, $\omega' = dg$ and so $M_{\gamma,2} = \int_{\gamma(t)} g \omega$. Developing this expression we get

$$M_{\gamma,2} = \sum_{1 \leq i < j \leq 3} W(a_i(t), a_j(t)) \int_{\gamma(t)} \phi_i d\phi_j. \quad (3.20)$$

Next Lemma is useful in following computations.

Lemma 3.8. *Assume that functions $f_i(x)$, $i = 1, \dots, m$, are holomorphic in some simply connected domain $U \subset \mathbb{C}_x$ containing the projection γ_x of γ to the x -axis \mathbb{C}_x . Then the iterated integral $\int_{\gamma} (f_1 dx) \dots (f_m dx)$ vanishes.*

Proof. This integral is equal to $\int_{\gamma_x} (f_1 dx) \dots (f_m dx)$, so this follows from Cauchy theorem. \square

By Lemma 3.8, $\int_{\gamma(t)} \phi_1 d\phi_3 = \int_{\gamma(t)} \log(x+1) \frac{dx}{x-1} \equiv 0$. On the other hand, since $M_{v_2,2} = W(a_2(t), a_1(t) - a_3(t)) \equiv 0$, we have $W(a_1, a_2) + W(a_2, a_3) = 0$, therefore

$$M_{\gamma,2} = W(a_1(t), a_2(t)) \left(\int_{\gamma(t)} \phi_1 d\phi_2 - \int_{\gamma(t)} \phi_2 d\phi_3 \right).$$

Substituting $\phi_2 d\phi_3$ by $-\phi_3 d\phi_2 + d(\phi_2 \phi_3)$, and since $\int_{\gamma(t)} d(\phi_2 \phi_3) = 0$, we have

$$M_{\gamma,2} = W(a_1(t), a_2(t)) \left(\int_{\gamma(t)} \phi_1 d\phi_2 + \phi_3 d\phi_2 \right).$$

Recall that $\phi_1 = \log(x+1)$, $\phi_2 = \log(y-1)$ and $\phi_3 = \log(x-1)$, so

$$\begin{aligned} M_2(\gamma(t)) &= W(a_1(t), a_2(t)) \int_{\gamma(t)} (\log(x+1) + \log(x-1)) \frac{dy}{y-1} \\ &= W(a_1(t), a_2(t)) \int_{\gamma(t)} \log(x^2-1) \frac{dy}{y-1} \\ &= W(a_1(t), a_2(t)) \int_{\gamma(t)} \log\left(\frac{t}{y^2-1}\right) \frac{dy}{y-1}. \end{aligned}$$

Again, by Lemma 3.8 this integral vanishes. \square

Proof of Proposition 3.2. Proposition 3.2 follows from Lemmas 3.6, 3.7. \square

3.4.1. $M_{\gamma,2} = M_{\gamma,3} = 0$ implies center.

Proposition 3.9. *For deformation (1.1) with ω as in (3.2), identical vanishing of both $M_{\gamma,2}, M_{\gamma,3}$ is equivalent to preservation of the center.*

One implication is trivial, so we assume $M_{\gamma,2} = M_{\gamma,3} \equiv 0$ and prove that (1.1) preserves the center.

First, there is a trivial symmetric case.

Lemma 3.10. *If either $a_2(t) \equiv 0$ or $a_1(t) - a_3(t) \equiv 0$, then (1.1) defines a center.*

Proof. Indeed, then the foliation (1.1) is symmetric with respect to the symmetry $y \rightarrow -y$ or with respect to the symmetry $x \rightarrow -x$, correspondingly. \square

Further, we assume that $a_1(t) - a_3(t), a_2(t) \neq 0$. The conditions $M_{\gamma,2} = M_{\gamma,3} = 0$ imply $M_{v_2,2} = M_{v_3,3} = 0$. From (3.14)(3.17), this is equivalent to

$$\begin{aligned} a_1 - a_3 &= \lambda_1 a_2, & \lambda_1 &\in \mathbb{C}^* \\ W(a_1, a_3) &= \lambda_2 a_2, & \lambda_2 &\in \mathbb{C}. \end{aligned} \quad (3.21)$$

This implies

$$W(a_1, a_2) = -a_2^2 \left(\frac{a_1}{a_2}\right)' = -\lambda a_2, \quad \lambda = \frac{\lambda_2}{\lambda_1}, \quad (3.22)$$

i.e. necessarily

$$\begin{aligned} a_1 &= a_2 \left(\lambda \int \frac{dt}{a_2} + c_1 \right) \\ a_3 &= a_2 \left(\lambda \int \frac{dt}{a_2} + c_1 - \lambda_1 \right). \end{aligned} \quad (3.23)$$

Denote $A(t) = \int \frac{dt}{a_2}$, so

$$\omega = a_2(t) [A(t)\alpha + d\phi], \quad \alpha = \lambda(d\phi_1 + d\phi_3), \quad \phi = c_1\phi_1 + \phi_2 + (c_1 - \lambda_1)\phi_3. \quad (3.24)$$

As $a_2(t) = A'(t)^{-1}$, the foliation (1.1) is then equal to

$$dF + \epsilon \frac{1}{A'(F)} (A(F)\alpha + d\phi) = 0, \quad (3.25)$$

which is orbitally equivalent to

$$dA(F) + \epsilon\eta = 0, \quad \eta = A(F)\alpha + d\phi. \quad (3.26)$$

If $\lambda = 0$, then $\alpha = 0$, and this is a Hamiltonian system with Hamiltonian $A(F) + \epsilon\phi$ (recall that ϕ is a holomorphic function in a neighborhood of γ). Thus $\lambda = 0$ implies preservation of center, and it remains to prove that vanishing of $M_{\gamma,3}$ implies $\lambda = 0$.

We consider (3.26) as a perturbation of a Hamiltonian system with Hamiltonian $A(F)$.

Lemma 3.11. *The first non-zero Melnikov functions $M_{\gamma,k}$ of (3.25) and the first non-zero Melnikov functions $\tilde{M}_{\gamma,k}$ of (3.26) are related by*

$$\tilde{M}_k = g'(F)M_k. \quad (3.27)$$

Proof. Indeed, the passage from (3.25) to (3.26) amounts to reparameterization of the transversal τ by values of $A(F)$. For a point $p \in \tau$ with $F(p) = t$,

$$\tilde{M}_{\gamma,k} = \frac{\partial A(F(P(\epsilon, p)))}{\partial \epsilon^k} = A'(t) \frac{\partial F(P(\epsilon, p))}{\partial \epsilon^k} = A'(t)M_{\gamma,k}. \quad \square$$

Remark 3.12. In other words, the first non-zero Melnikov function has tensor type of a vector field, which is expected from Proposition 3.4 and the following Remark.

Proof of Proposition 3.9. We will compute $\tilde{M}_{\gamma,3}$. By Françoise's algorithm [2, 3], we have that

$$\tilde{M}_{\gamma,3} = \int_{\gamma} (\eta' \eta)' \eta, \quad (3.28)$$

where the Gelfand-Leray derivative is taken with respect to the Hamiltonian $A(F)$. Developing the derivative,

$$\tilde{M}_{\gamma,3} = \int_{\gamma} \eta'' \eta \eta + \int_{\gamma} \eta' \eta' \eta + \int_{\gamma} \frac{\eta' \wedge \eta}{dA} \eta, \quad (3.29)$$

where $\eta' = \frac{d(A\alpha + d\phi)}{dA} = \alpha$, and $\eta'' = 0$, as the forms $\alpha, d\phi$ are closed. Then,

$$\tilde{M}_{\gamma,3} = \int_{\gamma} \alpha \alpha (A\alpha + d\phi) + \int_{\gamma} \frac{\alpha \wedge (A\alpha + d\phi)}{dA} \eta. \quad (3.30)$$

By Lemma 3.8 the triple integrals $\int_{\gamma} \alpha \alpha d\phi_1, \int_{\gamma} \alpha \alpha d\phi_3$ vanish. Also, as

$$\alpha = \lambda d \log F - \lambda d \log(y^2 - 1), \quad (3.31)$$

Lemma 3.8 implies that the integral $\int_{\gamma} \alpha \alpha d\phi_2$ vanishes.

Hence,

$$\tilde{M}_{\gamma,3} = \int_{\gamma} \frac{\alpha \wedge (A\alpha + d\phi)}{dA} \eta = \int_{\gamma} \frac{\alpha \wedge d\phi}{dA} \eta. \quad (3.32)$$

We have $\alpha \wedge d\phi = \alpha \wedge d\phi_2$. By (3.31), we have $\alpha \wedge d\phi_2 = \frac{\lambda dF}{F} \wedge d\phi_2$, so

$$\frac{\alpha \wedge d\phi}{dA} = \frac{\lambda \frac{dF}{F} \wedge d\phi_2}{A'(F)dF} = \frac{\lambda}{FA'(F)} d\phi_2, \quad (3.33)$$

and, as $\eta = A(F)\alpha + d\phi$,

$$\tilde{M}_{\gamma,3} = \frac{\lambda}{tA'(t)} \int_{\gamma} d\phi_2 \eta = \frac{\lambda}{A'(t)} \int_{\gamma} d\phi_2 \alpha + \frac{\lambda}{tA'(t)} \int_{\gamma} d\phi_2 d\phi. \quad (3.34)$$

Again, $\int_{\gamma} d\phi_2 \alpha \equiv 0$ by (3.31) and Lemma 3.8. Moreover,

$$\int_{\gamma} d\phi_2 d\phi = \int_{\gamma} d\phi_2 d\phi_2 + \frac{c_1}{\lambda} \int_{\gamma} d\phi_2 \alpha - \lambda \int_{\gamma} d\phi_2 d\phi_3 = -\lambda \int_{\gamma} d\phi_2 d\phi_3,$$

as $\int_{\gamma} d\phi_2 d\phi_2 \equiv 0$ by Lemma 3.8. Therefore

$$\tilde{M}_{\gamma,3} = \frac{-\lambda^2}{tA'(t)} \int_{\gamma} d\phi_2 d\phi_3. \quad (3.35)$$

We claim that

$$\int_{\gamma} d\phi_2 d\phi_3 \neq 0. \quad (3.36)$$

Indeed, from $\int_{\gamma} d\phi_i \equiv 0$ and $\int_{[\sigma_1, \sigma_2]} \omega_1 \omega_2 = \det \left\{ \int_{\sigma_i} \omega_j \right\}_{i,j=1}^2$, we see that $\int d\phi_2 d\phi_3$ vanishes on K , and, therefore, defines a linear functional on $(O/K)^*$. Therefore,

$$Var^2 \int_{\gamma} d\phi_2 d\phi_3 = \int_{v_2} d\phi_2 d\phi_3 = \det \begin{pmatrix} \int_{\delta_1+\delta_2} d\phi_2 & \int_{\delta_1+\delta_2} d\phi_3 \\ \int_{\delta_2+\delta_3} d\phi_2 & \int_{\delta_2+\delta_3} d\phi_3 \end{pmatrix} = 4\pi^2$$

by (3.1), which proves (3.36).

Thus $\tilde{M}_{\gamma,3}$, as well as $M_{\gamma,3}$, vanish identically only if $\lambda = 0$, which finishes the proof of Proposition 3.9. \square

Proof of Theorem 1.6. Take ω as in (1.7), with coefficients $a_i(F)$ as in Proposition 3.2. The first Melnikov function $M_{\gamma,1}$ vanishes identically for all ω of this type. Also, $M_{\gamma,2}$ vanishes by Proposition 3.2. Therefore $M_{\gamma,3}$ is the first non-zero Melnikov function of (1.1) or it is identically zero. In both cases, it is linear on the orbit, see [4, 7], and therefore $Var^3(M_{\gamma,3}(t)) = M_{v_3,3}(t) \neq 0$. Therefore $M_{\gamma,3}(t) \neq 0$, and, moreover, has length three. Taking $\alpha_1 = t$, $\alpha_2 = t^2$ and $c_0 = \lambda = 1$, one gets the example of Theorem 1.6.

The last statement of Theorem 1.6 follows from Proposition 3.9. \square

3.5. Length bigger than 4. Now, for deformation (1.1) consider the functions $M_{v_i,i}$, where v_i were defined in Proposition 2.7. Note that $M_{v_i,j}$, given by iterated integrals of length at most j , necessarily vanish on $v_i \in L_i$ for $j < i$, so $M_{v_i,i}$ are (generically) the first non-zero Melnikov functions of v_i with respect to the deformation (1.1).

Lemma 3.13. *The condition $M_{v_2,2} = M_{v_3,3} = 0$ implies $M_{v_i,i} = 0$ for all $i \geq 4$.*

Remark 3.14. Vanishing of $M_{v_i,i}$ is necessary for vanishing of $M_{\gamma,i}$ (i.e. follows from center conditions), but not sufficient, see for example Proposition 3.9.

Proof. Denote $\beta_1 = \int_{\delta_1+\delta_2} \omega$, $\beta_2 = \int_{\delta_2} \omega$ and $\beta_3 = \int_{\delta_2+\delta_3} \omega$. By Proposition 3.4,

$$\begin{aligned} M_{v_2,2} &= W(\beta_1, \beta_3) \\ M_{v_3,3} &= W(\beta_1, W(\beta_2, \beta_3)) \\ M_{v_4,4} &= W(\beta_1, W(\beta_2, W(\beta_2, \beta_3))) \\ &\vdots \\ M_{v_i,i} &= W(\beta_1, W(\beta_2, \dots, W(\beta_2, \beta_3)) \dots). \end{aligned}$$

Suppose $M_{v_2,2} \equiv 0$. If $\beta_3 \equiv 0$ then evidently all these Wronskians vanish. Otherwise, $\beta_1 = \lambda_1 \beta_3$, for some $\lambda \in \mathbb{C}$.

Suppose also that $M_3(v_3) \equiv 0$. Again, the case $\beta_1 \equiv 0$ is trivial. Otherwise, $W(\beta_2, \beta_3) = \lambda_2 \beta_1$, for some $\lambda_2 \in \mathbb{C}$, and therefore

$$W(\beta_2, \beta_3) = \lambda_1 \lambda_2 \beta_3,$$

which implies

$$M_{v_{i+1},i+1} = \lambda_1 \lambda_2 M_{v_i,i} = \dots = (\lambda_1 \lambda_2)^{i-2} M_{v_3,3} \equiv 0. \quad \square$$

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