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Toward universality in degree 2 of the Kricker lift of the Kontsevich integral and the Lescop equivariant invariant

Benjamin Audoux & Delphine Moussard

Abstract

In the setting of finite type invariants for null-homologous knots in rational homology 3–spheres with respect to null Lagrangian-preserving surgeries, there are two candidates to be universal invariants, defined respectively by Kricker and Lescop. In a previous paper, the second author defined maps between spaces of Jacobi diagrams. Injectivity for these maps would imply that Kricker and Lescop invariants are indeed universal invariants; this would prove in particular that these two invariants are equivalent. In the present paper, we investigate the injectivity status of these maps for degree 2 invariants, in the case of knots whose Blanchfield modules are direct sums of isomorphic Blanchfield modules of $\mathbb{Q}$–dimension two. We prove that they are always injective except in one case, for which we determine explicitly the kernel.

MSC: 57M27

Keywords: 3–manifold, knot, homology sphere, beaded Jacobi diagram, finite type invariant.

Contents

1 Introduction

The work presented here has its source in the notion of finite type invariants. This notion first appeared in independent works of Goussarov and Vassiliev involving invariants of knots in the 3–dimensional sphere S^3; in this case, finite type invariants are also called Vassiliev invariants. Finite type invariants of knots in S^3 are defined by their polynomial behaviour with respect to crossing changes. The discovery of the Kontsevich integral, which is a universal invariant among all finite type invariants of knots in S^3, revealed the depth of this class of invariants. It is known, for instance, that it dominates all Witten-Reshetikhin-Turaev quantum invariants. A theory of finite type invariants can be defined for any kind of topological objects provided that an elementary move on the set of these objects is fixed; the finite type invariants are defined by their polynomial behaviour with respect to this elementary move. For 3–dimensional manifolds, the notion of finite type invariants was introduced by Ohtsuki [Oht96], who constructed the first examples for integral homology 3–spheres, and it has been widely developed and generalized since then. In particular, Goussarov and Habiro independently developed a theory which involves any 3–dimensional manifolds—and their knots—and which contains the Ohtsuki theory for $\mathbb{Z}$–spheres [GGP01, Hab00]. In this case, the elementary move is the so-called Borromean surgery.

Garoufalidis and Rozansky introduced in [GR04] a theory of finite type invariants for knots in integral homology 3–spheres with respect to null-moves, which are Borromean surgeries satisfying a homological condition with respect to the knot. This theory was adapted to the "rational homology setting" by Lescop [Les13] who defined a theory of finite type invariants for null-homologous knots in rational homology 3–spheres with respect to null Lagrangian-preserving surgeries. In these theories, the degree 0 and 1 invariants are well understood and, up to them, there are two candidates to be universal finite type invariants, namely the Kricker rational lift of the Kontsevich integral [Kri00, GK04] and the Lescop equivariant invariant built from integrals over configuration spaces [Les11]. Both of them are known to be universal finite type invariants in two situations already: for knots in integral homology 3–spheres with trivial Alexander polynomial, with respect to null-moves [GR04], and for null-homologous knots in rational homology 3–spheres with trivial Alexander polynomial, with respect to null Lagrangian-preserving surgeries [Mou17]. In particular, the Kricker invariant and the Lescop invariant are equivalent for such knots—in the sense that they separate the same pairs of knots. Lescop conjectured in [Les13] that this equivalence holds in general.

Universal finite type invariants are known in other settings: the Kontsevich integral for links in S^3 [BN95], the Le–Murakami–Ohtsuki invariant and the Kontsevich–Kuperberg–Thurston invariant for integral homology 3–spheres [Le97] and for rational homology 3–spheres [Mou12a]. To establish universality of these invariants, the general idea is to give a combinatorial description of the graded space associated with the theory by identifying it with a graded space of diagrams. Such a project is developed in [Mou17] to study the universality of the Kricker and the Lescop invariants as finite type invariants of $\mathbb{Q}$*SK–pairs*, which are pairs made of a rational homology 3–sphere and a null-homologous knot in it.

Null Lagrangian-preserving surgeries preserve the Blanchfield module (defined over $\mathbb{Q}$), so one can reduce the study of finite type invariants of $\mathbb{Q}$SK–pairs to the set of $\mathbb{Q}$SK–pairs with a fixed Blanchfield module. In order to describe the graded space $\mathcal{G}(\mathfrak{A},\mathfrak{b})$ associated with finite type invariants of $\mathbb{Q}$SK–pairs with Blanchfield module $(\mathfrak{A},\mathfrak{b})$, a graded space of diagrams $\mathcal{A}^{aug}(\mathfrak{A},\mathfrak{b})$ is constructed in [Mou17], together with a surjective map $\varphi : \mathcal{A}^{aug}(\mathfrak{A},\mathfrak{b}) \to \mathcal{G}(\mathfrak{A},\mathfrak{b})$. Injectivity of this map would imply universality of the Kricker invariant and the Lescop invariant for $\mathbb{Q}$SK–pairs with Blanchfield module $(\mathfrak{A},\mathfrak{b})$ and consequently equivalence of these two invariants for such $\mathbb{Q}$SK–pairs.

Let $(\mathfrak{A},\mathfrak{b})$ be any Blanchfield module with annihilator $\delta \in \mathbb{Q}[t^{\pm1}]$. As detailed in [Mou17], we can focus on the subspace $\mathcal{G}^b(\mathfrak{A},\mathfrak{b}) = \oplus_{n\in\mathbb{Z}}\mathcal{G}^b_n(\mathfrak{A},\mathfrak{b})$ of $\mathcal{G}(\mathfrak{A},\mathfrak{b})$ associated with Borromean surgeries and study the restricted map $\varphi : \mathcal{A}(\mathfrak{A},\mathfrak{b}) \to \mathcal{G}^b(\mathfrak{A},\mathfrak{b})$ defined on a subspace $\mathcal{A}(\mathfrak{A},\mathfrak{b})$ of $\mathcal{A}^{aug}(\mathfrak{A},\mathfrak{b})$. Both the Lescop and the Kricker invariants are families $Z = (Z_n)_{n\in\mathbb{N}}$ of finite type invariants, where Z_n has degree n when n is even and Z_n is trivial when n is odd. For $\mathbb{Q}$SK–pairs with Blanchfield module $(\mathfrak{A},\mathfrak{b})$, Z_n takes values in a space $\mathcal{A}_n(\delta)$ of trivalent graphs with edges labelled in $\frac{1}{\delta}\mathbb{Q}[t^{\pm1}]$. The finiteness properties imply that Z_n induces a map on $\mathcal{G}^b_n(\mathfrak{A},\mathfrak{b})$. The map $\varphi : \mathcal{A}(\mathfrak{A},\mathfrak{b}) \to \mathcal{G}^b(\mathfrak{A},\mathfrak{b})$ decomposes as the direct sum of maps $\varphi_n : \mathcal{A}_n(\mathfrak{A},\mathfrak{b}) \to \mathcal{G}^b_n(\mathfrak{A},\mathfrak{b})$. Composing with Z_n, we get a map $\psi_n : \mathcal{A}_n(\mathfrak{A},\mathfrak{b}) \to \mathcal{A}_n(\delta)$; this provides the following commutative diagram:

$$\begin{tikzcd} & \mathcal{G}^b_n(\mathfrak{A},\mathfrak{b}) \arrow[dd, "Z_n"] \\ \mathcal{A}_n(\mathfrak{A},\mathfrak{b}) \arrow[ru, two heads, "\varphi_n"] \arrow[rd, "\psi_n"'] & \\ & \mathcal{A}_n(\delta) \end{tikzcd}$$

Note that the injectivity of ψ_n implies the injectivity of φ_n. When $(\mathfrak{A},\mathfrak{b})$ is a direct sum of N isomorphic Blanchfield modules, it has been established in [Mou17] that ψ_n is an isomorphism when $n \leq \frac{2}{3}N$. In particular, this applies for any $n \in \mathbb{N}$ when $(\mathfrak{A},\mathfrak{b})$ is the trivial Blanchfield module.

In this paper, we look into the case $n = 2$ when $(\mathfrak{A},\mathfrak{b})$ is a direct sum of N isomorphic Blanchfield modules of $\mathbb{Q}$-dimension two. According to the above-mentioned result, the map ψ_2 is then injective as soon as $N \geq 3$. The only remaining cases are hence $N = 1$ and $N = 2$. We prove the following (Propositions 4.7, 4.10 and 5.3):

Theorem 1.1. *If $(\mathfrak{A},\mathfrak{b})$ is a Blanchfield module of $\mathbb{Q}$-dimension two, with annihilator δ, then:*

1. *the map $\psi_2 : \mathcal{A}_2(\mathfrak{A},\mathfrak{b}) \to \mathcal{A}_2(\delta)$ is injective but not surjective;*
2. *the map $\psi_2 : \mathcal{A}_2(\mathfrak{A}\oplus\mathfrak{A},\mathfrak{b}\oplus\mathfrak{b}) \to \mathcal{A}_2(\delta)$ is injective if and only if $\delta \neq t + 1 + t^{-1}$; in this case, it is an isomorphism.*

It follows that, in degree 2, Kricker and Lescop invariants are indeed universal and equivalent for $\mathbb{Q}$SK–pairs with a Blanchfield module which is either of $\mathbb{Q}$–dimension two or a direct sum of isomorphic Blanchfield modules of $\mathbb{Q}$–dimension two, except in one exceptional case. But the most interesting, though unexpected, outcome of the above theorem is this latter exceptional case—namely the case of a Blanchfield module which is a direct sum of two isomorphic Blanchfield modules of order $t+1+t^{-1}$—for which the map ψ_2 is *not* injective. The kernel of ψ_2 in this situation is explicited in Proposition 4.10. A topological realization C is given in Figure 1: C is a linear combination of $\mathbb{Q}$SK-pairs whose class in $\mathcal{G}_2(\mathfrak{A},\mathfrak{b})$ is the image by φ_2 of a generator of the kernel of ψ_2. This leads to two alternatives. Either C has topological reasons to vanish in $\mathcal{G}_2(\mathfrak{A},\mathfrak{b})$, then the map φ_2 itself is not injective and some more efforts should be done to understand the combinatorial nature of $\mathcal{G}_n(\mathfrak{A},\mathfrak{b})$; or the Kricker and Lescop invariants do not induce, in general, injective maps on $\mathcal{G}_n^b(\mathfrak{A},\mathfrak{b})$, suggesting the existence of some yet unknown finite type invariants in this setting. In both cases, the discussion is recentered on the explicited counterexample which appears as a key example that should be studied further.

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2 Definitions and strategy

2.1 Definitions

Blanchfield modules. A *Blanchfield module* is a pair $(\mathfrak{A},\mathfrak{b})$ such that:

(i) $\mathfrak{A}$ is a finitely generated torsion $\mathbb{Q}[t^{\pm1}]$-module;

(ii) multiplication by $(1-t)$ defines an isomorphism of $\mathfrak{A}$;

(iii) $\mathfrak{b}:\mathfrak{A}\times\mathfrak{A}\to \mathbb{Q}(t)/_{\mathbb{Q}[t^{\pm1}]}$ is a non-degenerate hermitian form, *i.e.* $\mathfrak{b}(\eta,\gamma)(t)=\mathfrak{b}(\gamma,\eta)(t^{-1})$, $\mathfrak{b}(P(t)\gamma,\eta)=P(t)\mathfrak{b}(\gamma,\eta)$, and if $\mathfrak{b}(\gamma,\eta)=0$ for all $\eta\in\mathfrak{A}$, then $\gamma=0$.

Since $\mathbb{Q}[t^{\pm1}]$ is a principal ideal domain, there is a well-defined (up to multiplication by an invertible element of $\mathbb{Q}[t^{\pm1}]$) annihilator $\delta\in\mathbb{Q}[t^{\pm1}]$ for $\mathfrak{A}$. Condition (ii) implies that $\delta(1)\neq 0$ and Condition (iii) that δ is symmetric, *i.e.* $\delta(t^{-1})=\upsilon(t)\delta(t)$ with $\upsilon(t)$ invertible in $\mathbb{Q}[t^{\pm1}]$; see [Mou12b, Section 3.2] for more details. Moreover, it follows from $\mathfrak{b}$ being hermitian that $\mathfrak{b}(\gamma,\eta)\in\frac{1}{P}\mathbb{Q}[t^{\pm1}]$ if γ has order P.

In this paper, we focus on Blanchfield modules of $\mathbb{Q}$–dimension 2. In this case, either $\mathfrak{A}$ is cyclic, or it is a direct sum of two $\mathbb{Q}[t^{\pm1}]$–modules with the same order. In this latter case, it follows from δ being symmetric and $\delta(1)\neq 0$ that $\delta(t)=t+1$.

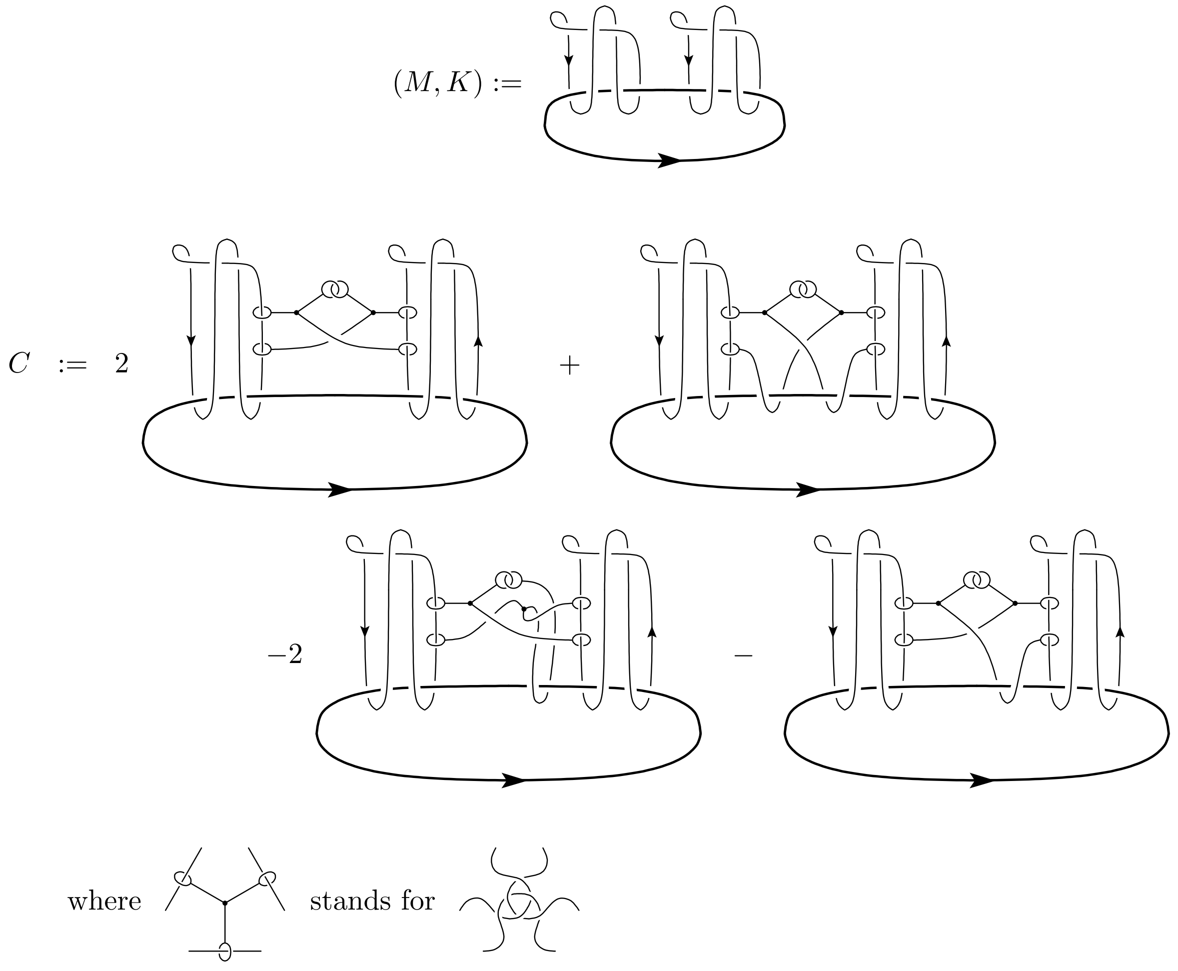


Figure 1: A topological realization for a generator of the kernel of ψ_2

Each picture represents the $\mathbb{Q}$SK–pair obtained by considering the copy of the thick unknot in the rational homology 3–sphere obtained by 0–surgery on the other two knots. The sum corresponds to the image by φ_2 of the generator of $\mathrm{Ker}(\psi_2)$ given in Proposition 4.10. There is indeed a correspondence between the four H–diagrams in the expression of this generator and the four terms in C, which are all of the form $(M, K)(T_1)(T_2)$ where T_1 and T_2 denote the two tripod graphs and $Y(T)$ denotes the result of the borromean surgery along T on Y. More precisely, each H–diagram is sent to $(M, K) - (M, K)(T_1) - (M, K)(T_2) + (M, K)(T_1)(T_2)$, but $(M, K)(T_1) = (M, K)(T_2) = (M, K)$. See [Mou12b] for the computation of the Alexander module of (M, K), [GGP01, Lemma 2.1] for the explicit action of the tripod graphs and [Mou17] for other definitions and details.

Spaces of $(\mathfrak{A},\mathfrak{b})$–colored diagrams. Fix a Blanchfield module $(\mathfrak{A},\mathfrak{b})$ and let $\delta \in \mathbb{Q}[t^{\pm 1}]$ be the annihilator of $\mathfrak{A}$. An $(\mathfrak{A},\mathfrak{b})$*–colored diagram* D is a uni-trivalent graph without strut (), given with:

- an orientation for each trivalent vertex, that is a cyclic order of the three half-edges that meet at this vertex;
- an orientation and a label in $\mathbb{Q}[t^{\pm 1}]$ for each edge;
- a label in $\mathfrak{A}$ for each univalent vertex;
- a rational fraction $f^D_{vv'}(t) \in \frac{1}{\delta}\mathbb{Q}[t^{\pm 1}]$ for each pair (v,v') of distinct univalent vertices of D, satisfying $f^D_{v'v}(t) = f^D_{vv'}(t^{-1})$ and $f^D_{vv'}(t)$ *mod* $\mathbb{Q}[t^{\pm 1}] = \mathfrak{b}(\gamma_v,\gamma_{v'})$, where γ_v and $\gamma_{v'}$ are the labels of v and v' respectively.

In the pictures, the orientation of trivalent vertices is given by  . When it does not seem to cause confusion, we write $f_{vv'}$ for $f^D_{vv'}$. We also call *legs* the univalent vertices. For $k \in \mathbb{N}$, we call *k–legs diagram* and *$k_{\leq}$–legs diagram* an $(\mathfrak{A},\mathfrak{b})$–colored diagram with, respectively, exactly and at most k legs. The *degree* of a colored diagram is the number of trivalent vertices of its underlying graph; the unique degree 0 diagram is the empty diagram.

The automorphism group $\mathrm{Aut}(\mathfrak{A},\mathfrak{b})$ of the Blanchfield module $(\mathfrak{A},\mathfrak{b})$ acts on $(\mathfrak{A},\mathfrak{b})$–colored diagrams by evaluation of an automorphism on the labels of all the legs of a diagram simultaneously. For $n \geq 0$, we set:

$$\mathcal{A}_n(\mathfrak{A},\mathfrak{b}) = \frac{\mathbb{Q}\big\langle (\mathfrak{A},\mathfrak{b})\text{–colored diagrams of degree } n\big\rangle}{\mathbb{Q}\big\langle \mathrm{AS}, \mathrm{IHX}, \mathrm{LE}, \mathrm{OR}, \mathrm{Hol}, \mathrm{LV}, \mathrm{EV}, \mathrm{LD}, \mathrm{Aut}\big\rangle},$$

where the relations AS (anti-symmetry), IHX, LE (linearity for edges), OR (orientation reversal), Hol (holonomy), LV (linearity for vertices), EV (edge-vertex) and LD (linking difference: this relation deals with the rational fractions associated to pairs of vertices) are described in Figure 2 and Aut is the set of relations $D = \zeta.D$ where D is a $(\mathfrak{A},\mathfrak{b})$–colored diagram and $\zeta \in \mathrm{Aut}(\mathfrak{A},\mathfrak{b})$. Since the opposite of the identity is an automorphism of $(\mathfrak{A},\mathfrak{b})$, we have $\mathcal{A}_{2n+1}(\mathfrak{A},\mathfrak{b}) = 0$ for all $n \geq 0$.

Spaces of δ–colored diagrams. Let $\delta \in \mathbb{Q}[t^{\pm 1}]$. A *$\delta$–colored diagram* is a trivalent graph whose vertices are oriented and whose edges are oriented and labelled by $\frac{1}{\delta}\mathbb{Q}[t^{\pm 1}]$. The *degree* of a δ–colored diagram is the number of its vertices. For every integer $n \geq 0$, set:

$$\mathcal{A}_n(\delta) = \frac{\mathbb{Q}\big\langle \delta\text{–colored diagrams of degree } n\big\rangle}{\mathbb{Q}\big\langle \mathrm{AS}, \mathrm{IHX}, \mathrm{LE}, \mathrm{OR}, \mathrm{Hol}, \mathrm{Hol}'\big\rangle},$$

where the relation Hol' is represented in Figure 3 and the relations AS, IHX, LE, OR, Hol are represented in Figure 2 but with edges now labelled in $\frac{1}{\delta}\mathbb{Q}[t^{\pm 1}]$. Note that in the case of $\mathcal{A}_n(\mathfrak{A},\mathfrak{b})$, the relation Hol' is induced by the relations Hol, EV, LD and LV, as shown in Figure 4, where LV is used to see that one diagram is trivial at each application of LD.

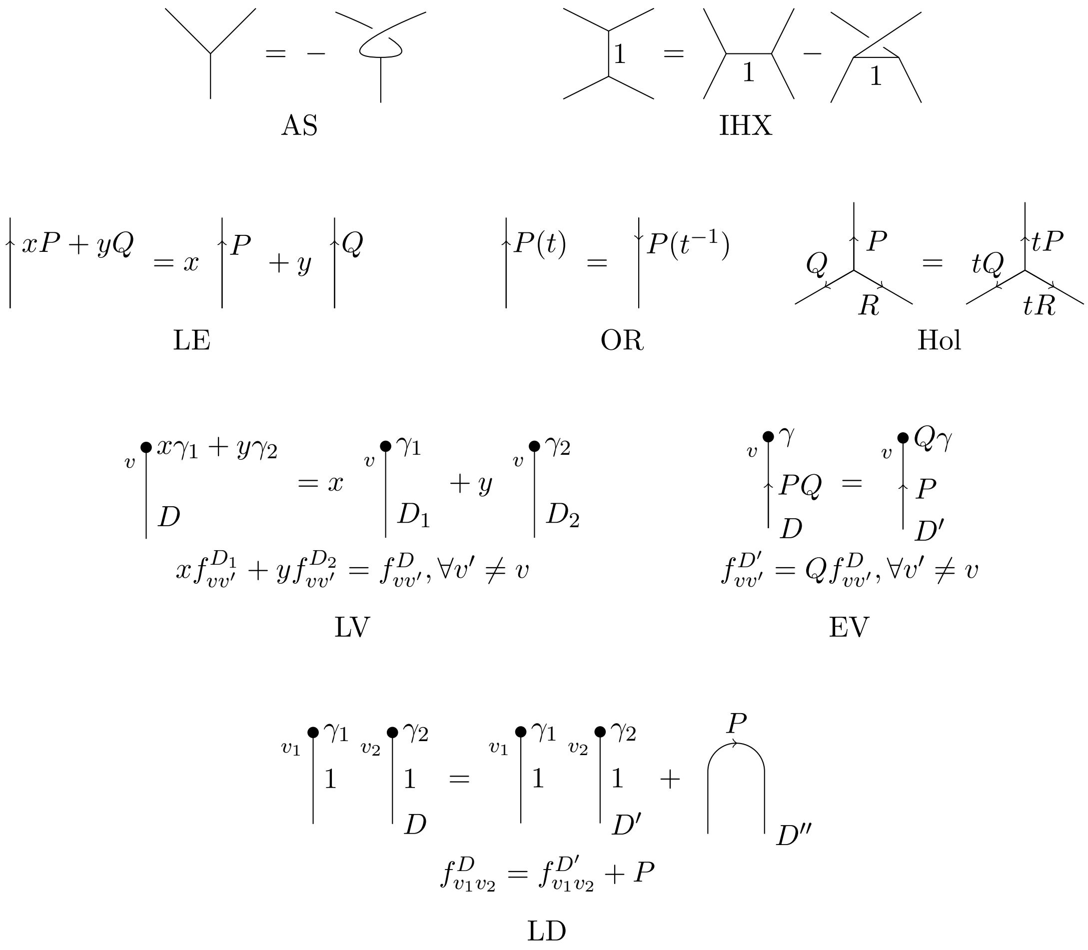


Figure 2: Relations on colored diagrams

In these pictures, $x, y \in \mathbb{Q}$, $P, Q, R \in \mathbb{Q}[t^{\pm 1}]$ and $\gamma, \gamma_1, \gamma_2 \in \mathfrak{A}$.

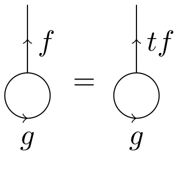


Figure 3: Relation Hol′

In this picture, $f, g \in \frac{1}{\delta}\mathbb{Q}[t^{\pm 1}]$.

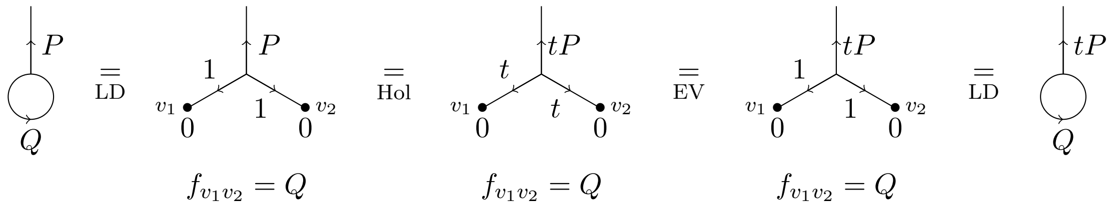


Figure 4: Recovering the relation Hol' from Hol, EV, LD and LV

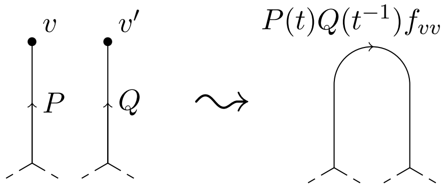


Figure 5: Pairing of two vertices

To an $(\mathfrak{A}, \mathfrak{b})$–colored diagram D of degree n, we associate a δ–colored diagram $\psi_n(D)$ as follows. Denote by V the set of legs of D. Define a *pairing of* V as an involution of V with no fixed point. For every such pairing p, define D_p as the diagram obtained by replacing, in D, every pair $\big(v, p(v)\big)$ of associated legs—and their adjacent edges—by a colored edge as indicated in Figure 5. Now set:

$$\psi_n(D) = \sum_{p \in \mathfrak{p}} D_p,$$

where $\mathfrak{p}$ is the set of pairings of V. Note that, if D has an odd number of legs, then $\mathfrak{p}$ is empty and $\psi_n(D) = 0$. One can easily check that this assignment yields a well-defined $\mathbb{Q}$-linear map $\psi_n : \mathcal{A}_n(\mathfrak{A}, \mathfrak{b}) \to \mathcal{A}_n(\delta)$.

2.2 Strategy

Getting rid of $\mathcal{A}_n(\delta)$. The map ψ_n involves two diagram spaces defined by different kind of diagrams, namely $(\mathfrak{A}, \mathfrak{b})$–colored diagrams and δ–colored diagrams. The following result will allow us to work with $(\mathfrak{A}, \mathfrak{b})$–colored diagrams only.

Theorem 2.1 ([Mou17, Theorem 2.12])**.** *Let n and N be non negative integers such that $N \geq \frac{3n}{2}$. Fix a Blanchfield module $(\mathfrak{A}, \mathfrak{b})$ with annihilator δ and define the Blanchfield module $(\mathfrak{A}, \mathfrak{b})^{\oplus N}$ as the direct sum of N copies of $(\mathfrak{A}, \mathfrak{b})$. Then δ is also the annihilator of $(\mathfrak{A}, \mathfrak{b})^{\oplus N}$ and the map $\overline{\psi}_n : \mathcal{A}_n\left((\mathfrak{A}, \mathfrak{b})^{\oplus N}\right) \to \mathcal{A}_n(\delta)$ is an isomorphism.*

This result provides a rewritting of the map ψ_n in the general case. There is indeed a natural map $\iota_n : \mathcal{A}_n(\mathfrak{A}, \mathfrak{b}) \to \mathcal{A}_n\left((\mathfrak{A}, \mathfrak{b})^{\oplus N}\right)$ defined on each diagram by interpreting the labels of its legs as elements of the first copy of $(\mathfrak{A}, \mathfrak{b})$ in $(\mathfrak{A}, \mathfrak{b})^{\oplus N}$, which makes the following diagram commute:

$$\begin{tikzcd}
 & \mathcal{A}_n\left((\mathfrak{A}, \mathfrak{b})^{\oplus N}\right) \arrow[dd, "\overline{\psi}_n", "\cong"'] \\
\mathcal{A}_n(\mathfrak{A}, \mathfrak{b}) \arrow[ur, "\iota_n"] \arrow[dr, "\psi_n"'] & \\
 & \mathcal{A}_n(\delta)
\end{tikzcd}.$$

In particular, the injectivity of ψ_n is equivalent to the injectivity of ι_n, what does not involve $\mathcal{A}_n(\delta)$ anymore. More generally, there is a natural map $\iota_n^{\ell} : \mathcal{A}_n\left((\mathfrak{A}, \mathfrak{b})^{\oplus \ell}\right) \to \mathcal{A}_n\left((\mathfrak{A}, \mathfrak{b})^{\oplus N}\right)$ defined similarly to ι_n. When it does not seem to cause confusion, ι_n^{ℓ} is simply denoted ι_n. When $n = 2$, for every $N \geq 3$, we have:

$$\begin{tikzcd}
\mathcal{A}_2(\mathfrak{A}, \mathfrak{b}) \arrow[r] \arrow[rr, bend left, "\iota_2^1"] \arrow[dr, "\psi_2^1"'] & \mathcal{A}_2\left((\mathfrak{A}, \mathfrak{b})^{\oplus 2}\right) \arrow[r, "\iota_2^2"] \arrow[d, "\psi_2^2"] & \mathcal{A}_2\left((\mathfrak{A}, \mathfrak{b})^{\oplus N}\right) \arrow[dl, "\cong"] \\
 & \mathcal{A}_2(\delta) &
\end{tikzcd}.$$

We focus on determining whether the maps ι_2^1 and ι_2^2 are injective or not. For that, it is sufficient to consider the case $N = 3$.

Filtration by the number of legs. The second point in our strategy is to consider the filtration induced by the number of legs. For $k = 0, \ldots, 3n$, let $\mathcal{A}_n^{(k)}(\mathfrak{A}, \mathfrak{b})$ be the subspace of $\mathcal{A}_n(\mathfrak{A}, \mathfrak{b})$ generated by $k_{\leq}$–legs diagrams and set:

$$\widehat{\mathcal{A}}_n^{(k)}(\mathfrak{A}, \mathfrak{b}) = \frac{\mathbb{Q}\big\langle k_{\leq}\text{–legs diagrams of degree } n\big\rangle}{\mathbb{Q}\langle \mathrm{AS}, \mathrm{IHX}, \mathrm{LE}, \mathrm{OR}, \mathrm{Hol}, \mathrm{LV}, \mathrm{EV}, \mathrm{LD}, \mathrm{Aut}\rangle}.$$

Recall that all these diagram spaces are trivial when n is odd. Moreover, in a uni-trivalent graph, the numbers of univalent and trivalent vertices have the same parity, thus $\mathcal{A}_{2n}^{(2k+1)}(\mathfrak{A}, \mathfrak{b}) = \mathcal{A}_{2n}^{(2k)}(\mathfrak{A}, \mathfrak{b})$ and $\widehat{\mathcal{A}}_{2n}^{(2k+1)}(\mathfrak{A}, \mathfrak{b}) \cong \widehat{\mathcal{A}}_{2n}^{(2k)}(\mathfrak{A}, \mathfrak{b})$. Obviously, $\widehat{\mathcal{A}}_n^{(3n)}(\mathfrak{A}, \mathfrak{b}) = \mathcal{A}_n(\mathfrak{A}, \mathfrak{b}) = \mathcal{A}_n^{(3n)}(\mathfrak{A}, \mathfrak{b})$. However, a subtlety of the structure of the spaces $\mathcal{A}_n(\mathfrak{A}, \mathfrak{b})$ is that the natural surjection $\widehat{\mathcal{A}}_n^{(k)}(\mathfrak{A}, \mathfrak{b}) \twoheadrightarrow \mathcal{A}_n^{(k)}(\mathfrak{A}, \mathfrak{b})$ is not, in general, an isomorphism. A counterexample is given in Proposition 4.1 (5.ii.), which underlies the case where ι_2^2 is not injective.

Reduction of the presentations. To study the injectivity status of the map ι_2, we first study the structure of the space $\mathcal{A}_2\left((\mathfrak{A}, \mathfrak{b})^{\oplus 3}\right)$ to determine if $\mathcal{A}_2^{(k)}\left((\mathfrak{A}, \mathfrak{b})^{\oplus 3}\right)$ is isomorphic to

$\widehat{\mathcal{A}}_2^{(k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right)$ for $k=2,4$. If we have such isomorphisms, then Corollary 3.5 states that the map ι_n is injective. Otherwise, we have to perform a similar study of the structure of $\mathcal{A}_2(\mathfrak{A},\mathfrak{b})$.

To understand the structures of these diagram spaces, the strategy is to simplify the given presentations by restricting simultaneously the set of generators and the set of relations. This reduction process is initialized in Section 3.2 for a general Blanchfield module and pursued in the next sections for each specific case.

3 Preliminary results

3.1 Distributed diagrams

We define notations that we will use throughout the rest of the paper. Let $(\mathfrak{A},\mathfrak{b})$ be a Blanchfield module with annihilator δ. For a positive integer N, set $(\mathfrak{A},\mathfrak{b})^{\oplus N}=\bigoplus_{i=1}^{N}(\mathfrak{A}_i,\mathfrak{b}_i)$, where each $(\mathfrak{A}_i,\mathfrak{b}_i)$ is an isomorphic copy of $(\mathfrak{A},\mathfrak{b})$, given with a fixed isomorphism $\xi_i:\mathfrak{A}\to\mathfrak{A}_i$ that respects the Blanchfield pairing. Define the *permutation automorphisms* ξ_{ij} of $(\mathfrak{A},\mathfrak{b})^{\oplus N}$ as $\xi_j\circ\xi_i^{-1}$ on $\mathfrak{A}_i$, $\xi_i\circ\xi_j^{-1}$ on $\mathfrak{A}_j$ and identity on the other $\mathfrak{A}_\ell$'s. Define Aut_ξ as the restriction of the Aut relation to these permutation automorphisms. Also denote by Aut_t and Aut_{-1} the restrictions of the Aut relation to the automorphisms that are the multiplication by t and -1 respectively on one $\mathfrak{A}_i$ and identity on the other $\mathfrak{A}_j$'s. If $(\mathfrak{A},\mathfrak{b})$ is cyclic, then define Aut_{res} as the union of Aut_ξ, Aut_t and Aut_{-1}. Otherwise, define Aut_{res} as the Aut relation restricted to permutation automorphisms and to automorphisms fixing one $\mathfrak{A}_i$ setwise and the others pointwise.

Finally, for $\ell\geq 0$, we say that an $\left((\mathfrak{A},\mathfrak{b})^{\oplus\ell}\right)$–colored diagram D is *distributed* if there is a partition of the legs of D into a disjoint union of pairs $\sqcup_{i\in I}\{v_i,w_i\}$ and an injective map $\sigma:I\to\{1,\dots,\ell\}$ such that the legs v_i and w_i are labelled in $\mathfrak{A}_{\sigma(i)}$ and the linking between vertices in different pairs is trivial.

Proposition 3.1 ([Mou17, Propositions 7.11 & 7.12]). *For all non negative integers n, k and ℓ such that $\ell\geq\frac{k}{2}$:*

$$\widehat{\mathcal{A}}_n^{(k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus\ell}\right)\cong\frac{\mathbb{Q}\Big\langle\text{distributed } k_{\leq}\text{–legs diagrams of degree } n\Big\rangle}{\mathbb{Q}\big\langle \mathrm{AS},\mathrm{IHX},\mathrm{LE},\mathrm{OR},\mathrm{Hol},\mathrm{LV},\mathrm{EV},\mathrm{LD},\mathrm{Aut}_{res}\big\rangle}.$$

In particular, for all integers $N\geq\frac{3n}{2}$:

$$\mathcal{A}_n\left((\mathfrak{A},\mathfrak{b})^{\oplus N}\right)\cong\frac{\mathbb{Q}\Big\langle\text{distributed }\left((\mathfrak{A},\mathfrak{b})^{\oplus N}\right)\text{–colored diagrams of degree } n\Big\rangle}{\mathbb{Q}\big\langle \mathrm{AS},\mathrm{IHX},\mathrm{LE},\mathrm{OR},\mathrm{Hol},\mathrm{LV},\mathrm{EV},\mathrm{LD},\mathrm{Aut}_{res}\big\rangle}.$$

For positive integers $\ell_1\leq\ell_2$, let $\widehat{\iota}_n:\widehat{\mathcal{A}}_n^{(k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus\ell_1}\right)\to\widehat{\mathcal{A}}_n^{(k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus\ell_2}\right)$ be the natural map defined on each diagram by interpreting the labels of its legs as elements of the first ℓ_1 copies of $(\mathfrak{A},\mathfrak{b})$ in $(\mathfrak{A},\mathfrak{b})^{\oplus\ell_2}$.

Corollary 3.2. *For all non negative integers n, k, ℓ_1 and ℓ_2 such that $\ell_1,\ell_2\geq\frac{k}{2}$, the map $\widehat{\iota}_n:\widehat{\mathcal{A}}_n^{(k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus\ell_1}\right)\to\widehat{\mathcal{A}}_n^{(k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus\ell_2}\right)$ is an isomorphism.*

Proof. A distributed $k_{\leq}$–legs diagram involves at most $2k$ copies of $\mathfrak{A}$; up to Aut_{ξ}, we can assume that these are copies whithin the first ℓ_1 ones. Conclude with Proposition 3.1. $\square$

The next lemma will be useful in particular to restrict the study of the map ι_2 to suitable quotients.

Corollary 3.3. *Let n, N, k and ℓ be non negative integers such that $N \geq \frac{3n}{2}$ and $\frac{k}{2} \leq \ell \leq N$. If $\mathcal{A}_n^{(k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus N}\right) \cong \widehat{\mathcal{A}}_n^{(k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus N}\right)$, then the map $\mathcal{A}_n^{(k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus \ell}\right) \to \mathcal{A}_n^{(k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus N}\right)$ induced by ι_n is an isomorphism.*

Proof. By Corollary 3.2, the map $\widehat{\iota}_n : \widehat{\mathcal{A}}_n^{(k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus \ell}\right) \to \widehat{\mathcal{A}}_n^{(k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus N}\right)$ is an isomorphism. Hence we have the following commutative diagram:

$$\begin{tikzcd}
\widehat{\mathcal{A}}_n^{(k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus \ell}\right) \arrow[dd] \arrow[dr, "\cong"] & \\
& \mathcal{A}_n^{(k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus N}\right) . \\
\mathcal{A}_n^{(k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus \ell}\right) \arrow[ur] &
\end{tikzcd}$$

The statement follows. $\square$

Lemma 3.4. *Let n, k, ℓ_1 and ℓ_2 be non negative integers such that $\ell_1 \leq \ell_2$ and $\frac{k}{2} \leq \ell_2$. Let $\widetilde{\mathcal{A}}_n^{(k)}$ denote the image of $\widehat{\mathcal{A}}_n^{(k)}$ in $\widehat{\mathcal{A}}_n^{(k+2)}$. Then the map $\widehat{\mathcal{A}}_n^{(k+2)}((\mathfrak{A},\mathfrak{b})^{\oplus \ell_1})/\widetilde{\mathcal{A}}_n^{(k)}((\mathfrak{A},\mathfrak{b})^{\oplus \ell_1}) \to \widehat{\mathcal{A}}_n^{(k+2)}((\mathfrak{A},\mathfrak{b})^{\oplus \ell_2})/\widetilde{\mathcal{A}}_n^{(k)}((\mathfrak{A},\mathfrak{b})^{\oplus \ell_2})$ induced by $\widehat{\iota}_n$ is injective.*

Proof. Let us define a left inverse of $\widehat{\iota}_n$. Let D be a distributed $(k+2)_{\leq}$–legs diagram. For each leg colored by $\eta \in \mathfrak{A}_i$ with $\ell_1 < i \leq \ell_2$, replace the label by $\xi_1 \circ \xi_i^{-1}(\eta)$. Choose any linkings coherent with these new labels. Thanks to the relation LD, any such choice defines the same class $\sigma_n(D)$ in the quotient $\widehat{\mathcal{A}}_n^{(k+2)}((\mathfrak{A},\mathfrak{b})^{\oplus \ell_1})/\widetilde{\mathcal{A}}_n^{(k)}((\mathfrak{A},\mathfrak{b})^{\oplus \ell_1})$. This provides a well-defined map $\sigma_n : \widehat{\mathcal{A}}_n^{(k+2)}((\mathfrak{A},\mathfrak{b})^{\oplus \ell_2})/\widetilde{\mathcal{A}}_n^{(k)}((\mathfrak{A},\mathfrak{b})^{\oplus \ell_2}) \to \widehat{\mathcal{A}}_n^{(k+2)}((\mathfrak{A},\mathfrak{b})^{\oplus \ell_1})/\widetilde{\mathcal{A}}_n^{(k)}((\mathfrak{A},\mathfrak{b})^{\oplus \ell_1})$ such that $\sigma_n \circ \widehat{\iota}_n = Id$. $\square$

Corollary 3.5. *Let n, ℓ and N be non negative integers such that n is even, $\ell \leq N$ and $N \geq \frac{3n}{2}$. If $\widehat{\mathcal{A}}_n^{(2k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus N}\right) \cong \mathcal{A}_n^{(2k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus N}\right)$ for all integers k such that $\ell \leq k \leq \frac{3n}{2}$, then the map $\iota_n : \mathcal{A}_n\left((\mathfrak{A},\mathfrak{b})^{\oplus \ell}\right) \to \mathcal{A}_n\left((\mathfrak{A},\mathfrak{b})^{\oplus N}\right)$ is injective. Moreover, it implies that $\widehat{\mathcal{A}}_n^{(2k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus \ell}\right) \cong \mathcal{A}_n^{(2k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus \ell}\right)$ for all $k \geq 0$.*

Proof. We prove by induction on k that $\widehat{\mathcal{A}}_n^{(2k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus \ell}\right) \cong \widetilde{\mathcal{A}}_n^{(2k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus \ell}\right) \cong \mathcal{A}_n^{(2k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus \ell}\right)$ and that the map $\mathcal{A}_n^{(2k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus \ell}\right) \to \mathcal{A}_n^{(2k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus N}\right)$ induced by ι_n is injective. For $k \leq \ell$, Corollary 3.2 says that $\widehat{\iota}_n : \widehat{\mathcal{A}}_n^{(2k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus \ell}\right) \to \widehat{\mathcal{A}}_n^{(2k)}\left((\mathfrak{A},\mathfrak{b})^{\oplus N}\right)$ is an isomorphism. For $k > \ell$, we use the following observation.

Fact. Let $f : E_1 \to E_2$ be a morphism between two vector spaces. Let $F_1 \subset E_1$ and $F_2 \subset E_2$ be linear subspaces such that $f(F_1) \subset F_2$ and let $\bar{f} : E_1/_{F_1} \to E_2/_{F_2}$ be the map induced by f. If $\bar{f}$ and $f_{|F_1}$ are injective, then f is injective.

Together with Lemma 3.4 and the induction hypothesis, this implies that the map $\widehat{\iota}_n : \widehat{\mathcal{A}}_n^{(2k)}\big((\mathfrak{A},\mathfrak{b})^{\oplus\ell}\big) \to \widehat{\mathcal{A}}_n^{(2k)}\big((\mathfrak{A},\mathfrak{b})^{\oplus N}\big)$ is injective. In both cases, we get the following commutative diagram:

$$\begin{tikzcd}
\widehat{\mathcal{A}}_n^{(2k)}\big((\mathfrak{A},\mathfrak{b})^{\oplus\ell}\big) \arrow[d, two heads] \arrow[dr, hook] & \\
\widetilde{\mathcal{A}}_n^{(2k)}\big((\mathfrak{A},\mathfrak{b})^{\oplus\ell}\big) \arrow[d, two heads] \arrow[r] & \mathcal{A}_n^{(2k)}\big((\mathfrak{A},\mathfrak{b})^{\oplus N}\big), \\
\mathcal{A}_n^{(2k)}\big((\mathfrak{A},\mathfrak{b})^{\oplus\ell}\big) \arrow[ur] &
\end{tikzcd}$$

which concludes the proof. ☐

3.2 First reduction of the presentations

Getting rid of lollipops. We start with a lemma on 0–labelled vertices.

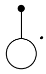
Lemma 3.6. *If D is an $(\mathfrak{A},\mathfrak{b})$–colored diagram with a 0–labelled vertex v, then*

$$D = \sum_{\substack{v' \text{ vertex of } D \\ v' \neq v}} D_{vv'},$$

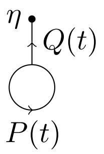
where $D_{vv'}$ is obtained from D by pairing v and v' as in Figure 5.

Proof. Since the vertex v is labelled by 0, the linking $f_{vv'}$ is a polynomial for any vertex $v' \neq v$. The conclusion follows using the relations LD and LV. ☐

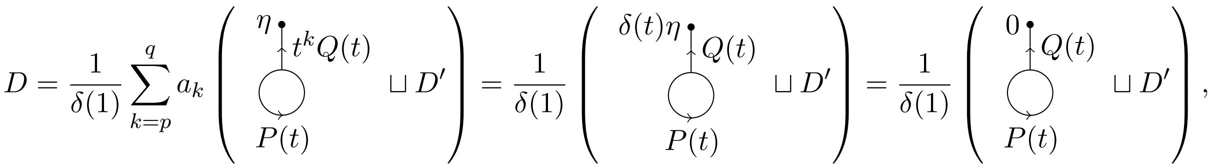
Now, the following lemma reduces the set of generators.

Lemma 3.7. *The general presentation of $\mathcal{A}_n(\mathfrak{A},\mathfrak{b})$ and the presentations of $\widehat{\mathcal{A}}_n^{(k)}\big((\mathfrak{A},\mathfrak{b})^{\oplus\ell}\big)$ and $\mathcal{A}_n\big((\mathfrak{A},\mathfrak{b})^{\oplus N}\big)$ given in Proposition 3.1 are still valid when removing from the generators the diagrams whose underlying graph contains a connected component* .

Proof. Thanks to the OR relation, such a diagram can be written

$D =$  $\sqcup D'.$

Writing $\delta = \sum_{k=p}^{q} a_k t^k$, we have:



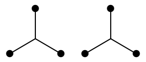
where the first equality holds since each diagram in the sum is equal to D by Hol$'$ and the second equality follows from EV and LV. Then, using Lemma 3.6, D can be written as a sum of diagrams with less legs. Check that all the relations involving D can be recovered from relations on diagrams with less legs. Conclude by decreasing induction on the number of legs. ☐

Finally, we state a corollary of Lemma 3.6 which will be useful later.

Corollary 3.8. *Let D be an $(\mathfrak{A}, \mathfrak{b})$–colored diagram and let v be a univalent vertex of D. If the annihilator of $\mathfrak{A}$ is $\delta = t + a + t^{-1}$, then*

$$D_+ = -aD - D_- + \sum_{\substack{v' \text{ vertex of } D \\ v' \neq v}} D_{vv'},$$

where D_+ and D_- are obtained from D by multiplying the label of v and the linkings $f_{vv'}$ by t and t^{-1} respectively, and $D_{vv'}$ is obtained from D by pairing v and v' as in Figure 5.

Taming 6 and 4–legs generators. We now give two lemmas that initialize the reduction process announced in Section 2.2. For that, define *YY–diagrams* similarly as $(\mathfrak{A}, \mathfrak{b})$–colored diagrams with underlying graph , except that edges are neither oriented nor labelled. Thanks to OR, those can be thought of as honest $(\mathfrak{A}, \mathfrak{b})$–colored diagrams with edges labelled by 1 and oriented arbitrarily. Define also $\overline{\text{Hol}}$ as the relations given in Figure 6; note that $\overline{\text{Hol}}$ is easily deduced from Hol and EV.

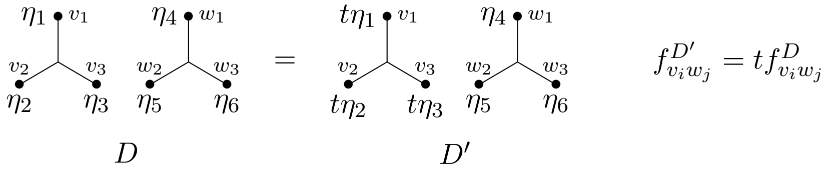
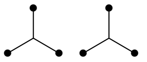
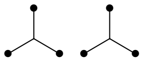


Figure 6: The relation $\overline{\text{Hol}}$

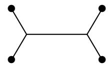
Lemma 3.9. *The space $\mathcal{A}_2(\mathfrak{A}, \mathfrak{b})$ admits the presentation with:*

- *as generators: YY–diagrams and all $4_{\leq}$–legs diagrams;*
- *as relations:* AS, LV, LD, Aut *and* $\overline{\text{Hol}}$ *on all generators and* IHX, LE, Hol, OR *and* EV *on $4_{\leq}$–legs generators.*

The space $\mathcal{A}_2\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)$ *admits the similar presentation with generators restricted to distributed* $\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)$*–colored diagrams and the relation* Aut *restricted to* Aut_{res}.

Proof. Any degree two $(\mathfrak{A},\mathfrak{b})$–colored diagram with six legs has underlying graph  . Using LE, any such diagram can be written as a $\mathbb{Q}$–linear combination of diagrams having all edges labelled by powers of t. Then, using OR and EV, these powers of t can be pushed to the legs. This produces a canonical decomposition of any 6–legs diagram in terms of YY–diagrams. Hence it provides a $\mathbb{Q}$–linear map from the $\mathbb{Q}$–vector space freely generated by all $(\mathfrak{A},\mathfrak{b})$–colored diagrams of degree 2 to the module $\mathcal{A}_2'(\mathfrak{A},\mathfrak{b})$ defined by the presentation given in the statement. This map descends to a well-defined map τ from $\mathcal{A}_2(\mathfrak{A},\mathfrak{b})$ to $\mathcal{A}_2'(\mathfrak{A},\mathfrak{b})$. Indeed, it is sufficient to check that all generating relation in $\mathcal{A}_2(\mathfrak{A},\mathfrak{b})$ is sent to zero. It is immediate for AS, LE, OR, LV, LD and Aut; it is true for EV and Hol by applying LV and $\overline{\mathrm{Hol}}$ respectively on the image; it also holds for IHX since there is no such relation involving diagrams with underlying graph  .

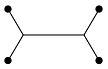
Now, it is clear that sending a diagram to itself gives a well-defined map $\mathcal{A}_2'(\mathfrak{A},\mathfrak{b}) \to \mathcal{A}_2(\mathfrak{A},\mathfrak{b})$ which is the inverse of τ. □

Now, we address the case of 4–legs generators. For that, we define *H–diagrams* similarly as $(\mathfrak{A},\mathfrak{b})$–colored diagrams with underlying graph  , except that edges are neither oriented nor labelled. Again, thanks to OR, those can be thought of as honest $(\mathfrak{A},\mathfrak{b})$–colored diagrams with edges labelled by 1 and oriented arbitrarily.

Lemma 3.10. *The space* $\widehat{\mathcal{A}}_2^{(4)}(\mathfrak{A},\mathfrak{b})$ *admits the presentation with:*

- *as generators: H–diagrams and all* $2_{\leq}$*–legs diagrams;*
- *as relations:* AS*,* IHX*,* LV*,* LD *and* Aut *on all generators and* LE*,* Hol*,* OR *and* EV *on* $2_{\leq}$*–legs generators.*

The space $\widehat{\mathcal{A}}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)$ *admits the similar presentation with generators restricted to distributed* $\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)$*–colored diagrams and the relation* Aut *restricted to* Aut_{res}.

Proof. First use Lemma 3.7 to reduce the 4–legs generators to those with underlying graph  and then proceed as in the previous lemma. Here, the relation Hol is also needed to remove the power of t from the central edge and the obtained decomposition is not anymore canonical. However, two possible decompositions are related by the relation of Aut associated with the automorphism that multiplies the whole Blanchfield module by t. □

Taming leg labels. Now, we want to go further in the reduction of the presentations. Fix a $\mathbb{Q}$-basis ω of $\mathfrak{A}$. For all $\gamma,\eta\in\omega$, fix $f(\gamma,\eta)\in\mathbb{Q}(t)$ such that $\mathfrak{b}(\gamma,\eta)=f(\gamma,\eta)$ *mod* $\mathbb{Q}[t^{\pm1}]$. For $\ell\geq 1$, identify $(\mathfrak{A},\mathfrak{b})^{\oplus\ell}$ with $\oplus_{1\leq i\leq\ell}(\mathfrak{A}_i,\mathfrak{b}_i)$ and let Ω be the union of the $\xi_i(\omega)$ for $i=1,\dots,\ell$. An $(\mathfrak{A},\mathfrak{b})^{\oplus\ell}$–colored diagram (resp. YY–diagram, H–diagram) is called *ω–admissible*, or simply *admissible* when there is no ambiguity on ω, if:

(i) its legs are colored by elements of Ω,

(ii) for two vertices v and w that are respectively colored by $\xi_i(\gamma)$ and $\xi_j(\eta)$, $f_{vw} = f(\gamma, \eta)$ if $i = j$ and $f_{vw} = 0$ otherwise.

Every $(\mathfrak{A}, \mathfrak{b})^{\oplus \ell}$–colored diagram (resp. YY–diagram, H–diagram) D has a canonical *ω–reduction*, which is the decomposition as a $\mathbb{Q}$–linear sum of ω–admissible diagrams obtained as follows. Write all the labels of the legs as $\mathbb{Q}$–linear sums of elements of Ω. Then use LV to write D as a $\mathbb{Q}$–linear sum of diagrams with legs labelled by $\Omega \cup \{0\}$ and the Ω-labelled legs satisfying Condition (ii). Finally, apply repeatedly Lemma 3.6 to remove 0–labelled vertices.

In the next step, we will not be able to reduce further the sets of generators and relations without rewriting some of the relations first. Denote by Aut^ω the set of relations $D = \Sigma$ where D is an ω–admissible diagram and Σ is the ω–reduction of $\zeta.D$ for $\zeta \in \mathrm{Aut}(\mathfrak{A}, \mathfrak{b})$. Define similarly $\mathrm{Aut}^\omega_{res}$ and Aut^ω_t. Define $\overline{\mathrm{Hol}}^\omega$ as the set of relations that identify an ω–admissible diagram D with the ω–reduction of the corresponding diagram D' of Figure 3.

In general, if a family of generators is given for the group $\mathrm{Aut}(\mathfrak{A}, \mathfrak{b})$, then the Aut relations, as well as the Aut^ω relations, can be restricted to the set of relations provided by the automorphisms of this generating family.

Lemma 3.11. *The space $\mathcal{A}_2(\mathfrak{A}, \mathfrak{b})$ admits the presentation with:*

- *as generators: ω–admissible YY–diagrams and all $4_\leq$–legs diagrams;*
- *as relations:* AS, Aut^ω *and* $\overline{\mathrm{Hol}}^\omega$ *on 6–legs generators and* AS, IHX, Hol, LE, OR, LV, LD, EV *and* Aut *on $4_\leq$–legs generators.*

The space $\mathcal{A}_2\big((\mathfrak{A}, \mathfrak{b})^{\oplus 3}\big)$ admits the similar presentation with generators restricted to distributed $\big((\mathfrak{A}, \mathfrak{b})^{\oplus 3}\big)$–colored diagrams and the relations Aut^ω *restricted to* $\mathrm{Aut}^\omega_{res}$*. If $\mathfrak{A}$ is cyclic,* $\mathrm{Aut}^\omega_{res}$ *can be replaced by the union of* Aut_ξ *and* Aut^ω_t*.*

Proof. Starting from the presentation given in Lemma 3.9 and using the ω–reduction, one can proceed as in the proof of Lemma 3.9. The only difficulty is to prove that the ω–reduction of all Aut and $\overline{\mathrm{Hol}}$ relations are indeed zero in the new presentation. To see that for Aut, consider a relation $D = \zeta.D$ for an $(\mathfrak{A}, \mathfrak{b})$–colored diagram D and an automorphism $\zeta \in \mathrm{Aut}(\mathfrak{A}, \mathfrak{b})$. Let $D = \sum_i \alpha_i D_i$ be the ω–reduction of D. For each i, write $\zeta.D_i = \sum_s \beta^i_s D^i_s$ the ω–reduction of the diagram $\zeta.D_i$. Check that $\zeta.D = \sum_i \alpha_i \sum_s \beta^i_s D^i_s$ is the ω–reduction of $\zeta.D$. It follows that the relation $D = \zeta.D$ is sent onto a $\mathbb{Q}$–linear combination of the relations $D_i = \sum_s \beta^i_s D^i_s$, which are in Aut^ω. Relations $\overline{\mathrm{Hol}}$ can be handled similarly.

For the last assertion, note that the relation Aut_ξ never identifies an admissible diagram with a non-admissible one and that the relation Aut_{-1} on admissible distributed diagrams only induces trivial relations. □

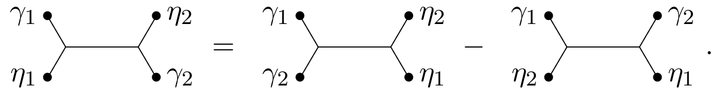
For the reduction of the 4–legs generators, we focus on the $(\mathfrak{A}, \mathfrak{b})^{\oplus 3}$ case and we introduce a more restrictive notion of admissible diagrams. An ω–admissible H–diagram is *strongly ω–admissible*, or simply *strongly admissible* when there is no ambiguity on ω, if its legs are colored in $\mathfrak{A}_1$ and $\mathfrak{A}_2$ and if two legs adjacent to a same trivalent vertex are labelled in different $\mathfrak{A}_i$'s.

Lemma 3.12. *The space $\widehat{\mathcal{A}}^{(4)}_2\big((\mathfrak{A}, \mathfrak{b})^{\oplus 3}\big)$ admits the presentation with:*

- *as generators: strongly ω–admissible H–diagrams and all $2_{\leq}$–legs diagrams;*
- *as relations:* AS *and* $\mathrm{Aut}^{\omega}_{res}$ *on 4–legs generators and* AS, IHX, LE, Hol, OR, LV, LD, EV *and* Aut *on $2_{\leq}$–legs generators.*

If $\mathfrak{A}$ is cyclic, $\mathrm{Aut}^{\omega}_{res}$ can be replaced by the union of Aut_{ξ} and $\mathrm{Aut}^{\omega}_{t}$.

Proof. Via at most one Aut_{ξ} relation, any ω–admissible H–diagram is equal to an ω–admissible H–diagram whose legs are labelled by $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Moreover, if $\gamma_1, \eta_1 \in \mathfrak{A}_1$ and $\gamma_2, \eta_2 \in \mathfrak{A}_2$, then the IHX relation gives:



It follows that any H–diagram has a canonical decomposition in terms of strongly ω–admissible H–diagrams. Proceed then as in the proof of Lemma 3.11. □

A set $\mathcal{E}$ of ω–admissible YY–diagrams (resp. H–diagrams) is *essential* if any ω–admissible YY–diagram (resp. H–diagram) which is not in $\mathcal{E}$ is either equal to a diagram in $\mathcal{E}$ via an AS or Aut_{ξ} relation, or trivial by AS. Denote by $\mathrm{Aut}^{\mathcal{E}}$ the set of relations $D = \Sigma$, where D is an element of $\mathcal{E}$ and Σ is the ω–reduction of $\zeta.D$ for some $\zeta \in \mathrm{Aut}(\mathfrak{A}, \mathfrak{b})$, rewritten in terms of $\mathcal{E}$. Define similarly $\overline{\mathrm{Hol}}^{\mathcal{E}}$ and $\mathrm{Aut}^{\mathcal{E}}_{*}$, where Aut_{*} is any subfamily of Aut described as the relations arising from the action of a subset of $\mathrm{Aut}(\mathfrak{A}, \mathfrak{b})$—for instance Aut_{res} or Aut_{t}.

Lemma 3.13. *If $\mathcal{E}$ is an essentiel set of ω–admissible YY–diagrams (resp. H–diagrams), then the YY–diagrams (resp. H–diagrams) in the set of generators of the presentation given in Lemma 3.11 (resp. Lemma 3.12) can be restricted to $\mathcal{E}$ and the relations Aut^{ω}, $\mathrm{Aut}^{\omega}_{res}$, $\mathrm{Aut}^{\omega}_{t}$ and $\overline{\mathrm{Hol}}^{\omega}$ can be replaced by $\mathrm{Aut}^{\mathcal{E}}$, $\mathrm{Aut}^{\mathcal{E}}_{res}$, $\mathrm{Aut}^{\mathcal{E}}_{t}$ and $\overline{\mathrm{Hol}}^{\mathcal{E}}$ respectively. Moreover, if $\mathcal{E}$ is minimal, then* AS *and* Aut_{ξ} *on YY–diagrams (resp. H-diagrams) can be removed from the set of relations.*

Proof. If an ω–admissible diagram is trivial by AS, then a relation $\overline{\mathrm{Hol}}$ or Aut involving this diagram gives a trivial relation; indeed, the terms in the corresponding decomposition are trivial or cancel by pairs. Similarly, if two ω–admissible diagrams are related by a relation AS, then the relations $\overline{\mathrm{Hol}}$ and Aut applied to these diagrams provide the same relations.

If D is an ω–admissible diagram and $D' = \xi_{ij}.D$ for some permutation automorphism ξ_{ij}, then any $\overline{\mathrm{Hol}}$ relation involving D' is recovered from the action of ξ_{ij} on the corresponding $\overline{\mathrm{Hol}}$ relation involving D, and the relation resulting from the action of some automorphism ζ on D' is recovered by the action of $\xi_{ij} \circ \zeta \circ \xi_{ij}$ on D.

For the last assertion, it is sufficient to notice that an AS relation makes either two generators to be equal, or a generator to be trivial, and that an Aut_{ξ} relation always identifies two generators. □

At this point, we have reduced the presentation for $\mathcal{A}_2(\mathfrak{A}, \mathfrak{b})$ so that we only have to consider non (AS and Aut_{ξ}–trivially) redundant YY–diagrams with prescribed rational fractions on pairs of vertices depending only on the labels, which are all in a given $\mathbb{Q}$–basis of $\mathfrak{A}$, and $4_{\leq}$–legs diagrams; the YY–diagrams being only subject to Aut and $\overline{\mathrm{Hol}}$ relations rewritten in these YY–diagrams.

A similar reduction has been done for $\mathcal{A}_2\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)$, where Aut is even replaced by Aut_{res}; if $\mathfrak{A}$ is cyclic, the latter can further be replaced by Aut_t. Likewise, the presentation for $\widehat{\mathcal{A}}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)$ has been reduced so that we only have to consider non (AS and Aut_ξ–trivially) redundant H–diagrams with prescribed rational fractions on pairs of vertices depending only on the labels, which are all in a given $\mathbb{Q}$–basis of $\mathfrak{A}$, and $2_\le$–legs diagrams; the H–diagrams being only subject to Aut_{res} relations rewritten in these H–diagrams; if $\mathfrak{A}$ is cyclic, Aut_{res} can further be replaced by Aut_t.

4 Case when $\mathfrak{A}$ is of $\mathbb{Q}$–dimension two and cyclic

In this section, we assume that $\mathfrak{A}$ is a cyclic Blanchfield module of $\mathbb{Q}$–dimension two. Let $\delta = t + a + t^{-1}$ be its annihilator; note that $a \neq -2$. Let γ be a generator of $\mathfrak{A}$. Since the pairing $\mathfrak{b}$ is hermitian and non degenerate, we can set $\mathfrak{b}(\gamma,\gamma) = \frac{r}{\delta}$ *mod* $\mathbb{Q}[t^{\pm 1}]$ with $r \in \mathbb{Q}^*$. Throughout this section, we fix the basis ω to be $\{\gamma, t\gamma\}$ and we set $f(t^{\varepsilon_1}\gamma, t^{\varepsilon_2}\gamma) = t^{\varepsilon_1-\varepsilon_2}\frac{r}{\delta}$, where $\varepsilon_1, \varepsilon_2 \in \{0,1\}$. Accordingly, set $\gamma_i = \xi_i(\gamma)$ for $i = 1,2,3$.

4.1 Structure of $\mathcal{A}_2\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right)$

The main results of this section are gathered in the following proposition.

Proposition 4.1. *If $(\mathfrak{A},\mathfrak{b})$ is a cyclic Blanchfield module of $\mathbb{Q}$–dimension two with annihilator $t + a + t^{-1}$, then:*

1. $\mathcal{A}_2^{(2)}\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right) \cong \widehat{\mathcal{A}}_2^{(2)}\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right)$*;*
2. $\mathcal{A}_2\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right)/\mathcal{A}_2^{(2)}\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right)$ *is freely generated by the diagrams H_1 and G_1 of Figure 7;*
3. *the natural map* $\widehat{\mathcal{A}}_2^{(2)}\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right) \to \widehat{\mathcal{A}}_2^{(4)}\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right)$ *is injective and the corresponding quotient* $\widehat{\mathcal{A}}_2^{(4)}\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right)/\widehat{\mathcal{A}}_2^{(2)}\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right)$ *is freely generated by the H–diagrams H_1 and H_3 given in Figure 7;*
4. *if $a \neq 1$, then* $\mathcal{A}_2\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right) = \mathcal{A}_2^{(4)}\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right) \cong \widehat{\mathcal{A}}_2^{(4)}\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right)$*;*
5. *if $a = 1$, then*
   - *i.* $\mathcal{A}_2^{(4)}\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right) \subsetneq \mathcal{A}_2\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right)$ *and the quotient* $\mathcal{A}_2^{(4)}\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right)/\mathcal{A}_2^{(2)}\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right)$ *is freely generated by the H–diagram H_1 given in Figure 10;*
   - *ii.* $\mathcal{A}_2^{(4)}\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right) \not\cong \widehat{\mathcal{A}}_2^{(4)}\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right)$.

The proof of this proposition will derive from the next results, which resume the reduction process where it was left at the end of Section 3.2. In order to make the text easier, we will denote by Aut_t^i, for any $i \in \{1,2,3\}$, the Aut_t relation applied on $\mathfrak{A}_i$.

Lemma 4.2. *The space $\mathcal{A}_2\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)$ admits the presentation with:*

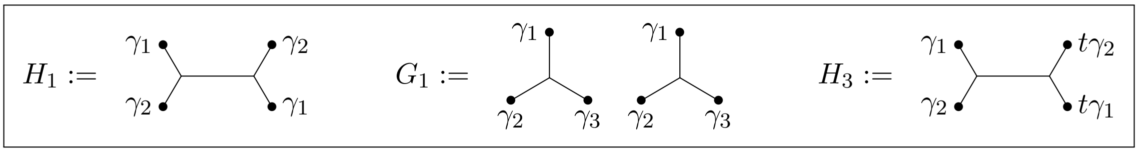


Figure 7: Some generators for our diagram spaces

In these pictures, all edges are labelled by 1 and the linkings are given by $f_{vw} = r/\delta$ when v and w are labelled by the same γ_i and 0 otherwise.

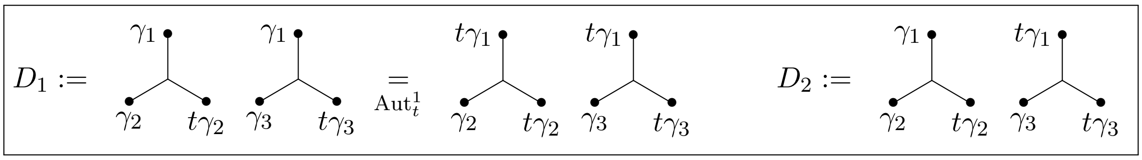


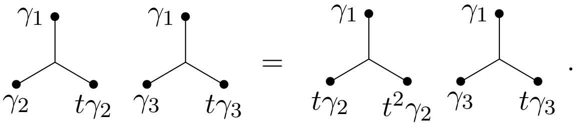
Figure 8: First family of 6–legs generators

- *as generators: the YY–diagrams D_1, D_2 of Figure 8 and G_1, G_2, G_3, G_4 of Figure 9 and all $4_{\leq}$–legs diagrams;*
- *as relations:* AS, IHX, LE, Hol, OR, LV, LD, EV *and* Aut *on $4_{\leq}$–legs generators and the following relations, where H_1, H_2, H_3, H_4 are the H–diagrams given in Figure 10:*

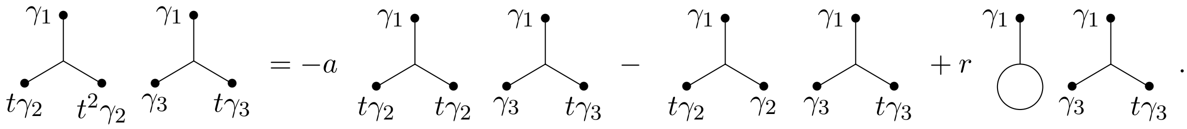
$$\begin{cases} D_1 = D_2 \\ (a+2)D_1 = r(H_3 - H_4) \\ aG_1 + 2G_2 = rH_1 \\ G_1 + aG_2 + G_4 = rH_3 \\ aG_3 + 2G_4 = rH_4 \\ (a+1)G_2 + G_3 = rH_2 \end{cases}.$$

Proof. Thanks to Lemmas 3.11 and 3.13, we only have to check that the relations $\overline{\mathrm{Hol}}$ and Aut_t applied to the admissible diagrams of Figures 8 and 9 give exactly the six new relations.

We begin with the first family. Applying Aut_t^2 to D_1, we obtain:



By Corollary 3.8, we have:



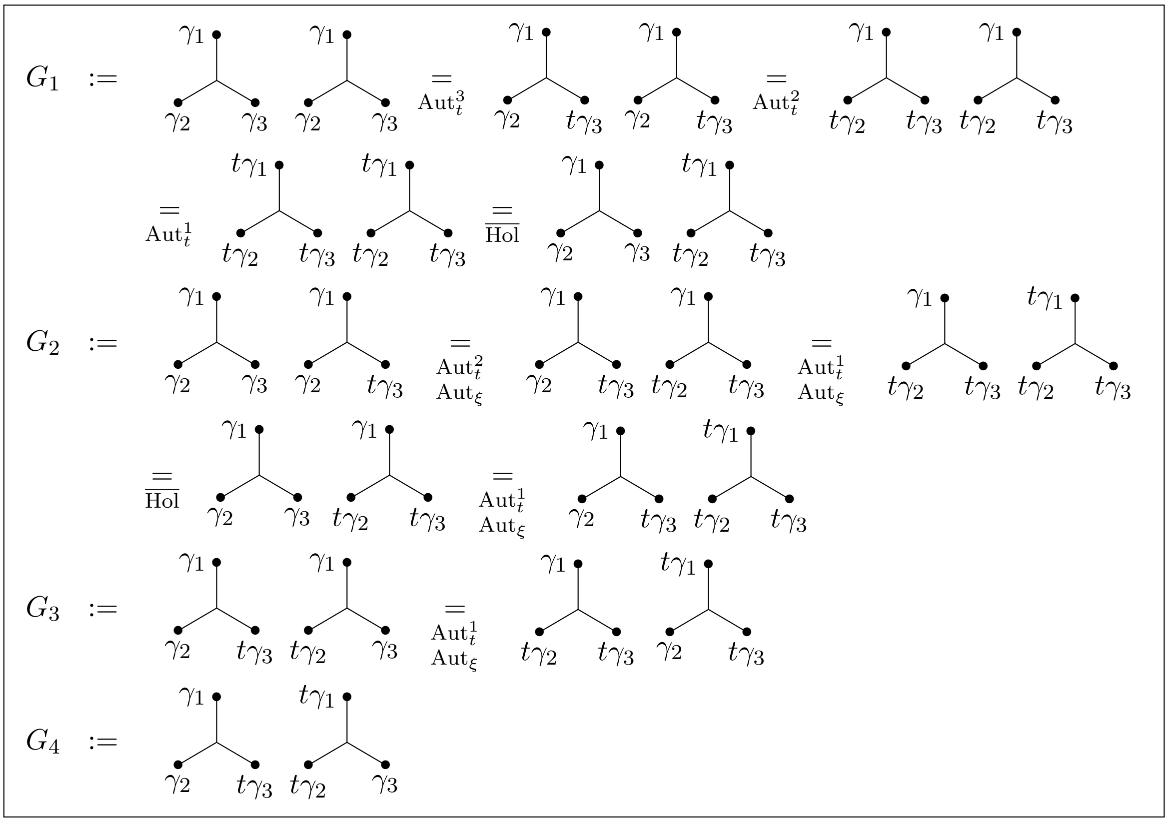


Figure 9: Second family of 6–legs generators

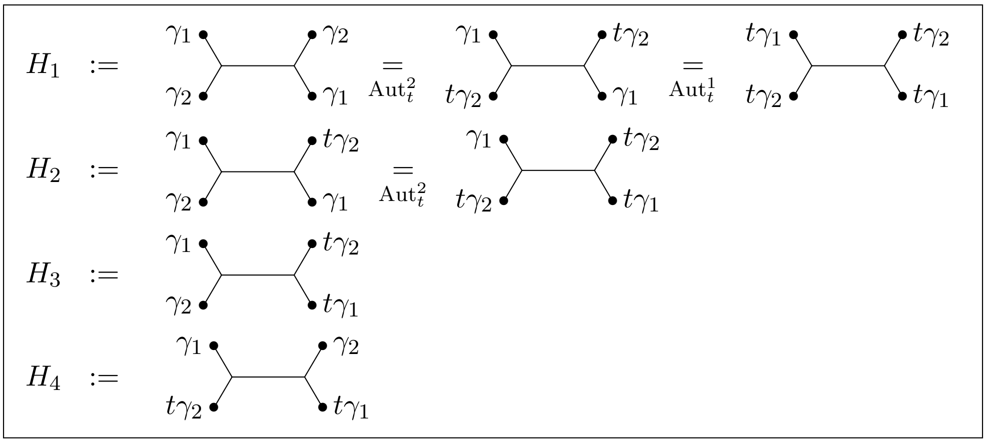
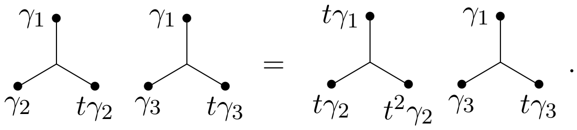


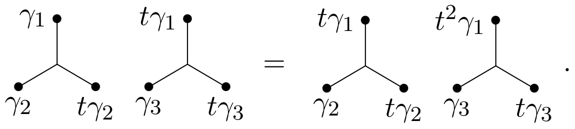
Figure 10: Family of 4–legs generators

In this equality, the second and fourth diagrams are trivial by AS and we get $D_1 = D_1$. Application of Aut_t^3 to D_1 is similar and gives the same result. Now, applying the $\overline{\mathrm{Hol}}$ relation to D_1, we obtain:

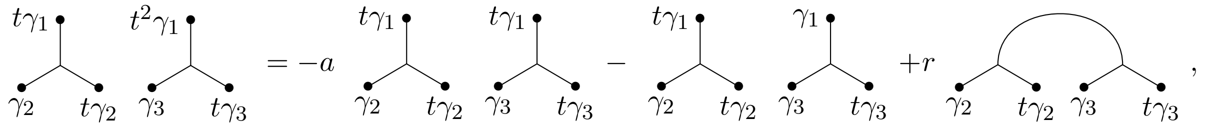


Applying Corollary 3.8 as previously, we get $D_1 = D_2$. One can check that applying $\overline{\mathrm{Hol}}$ and Aut_t to the second form of D_1 does not give any additional relation.

We now have to apply the same relations to D_2. Applying Aut_t^1 to D_2 gives:

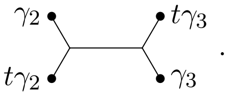


Once again we use Corollary 3.8 to get:



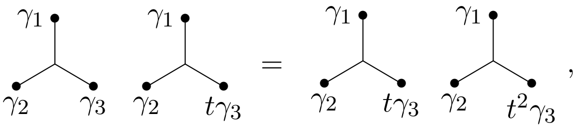
and finally:

$$D_1 = \frac{r}{a+2}$$

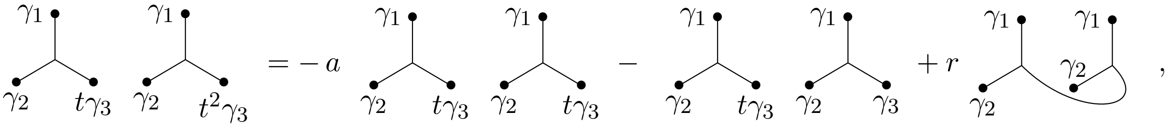


One can check that applying the other Aut_t or the $\overline{\mathrm{Hol}}$ relations to D_2 does not give any additional relation.

We turn to the second family of 6–legs generators. Applying Aut_t^3 to G_2 gives:

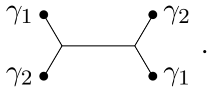


and by Corollary 3.8, we have:

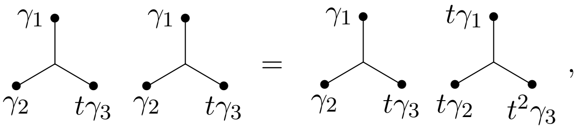


so we get the relation:

$$aG_1 + 2G_2 = r$$

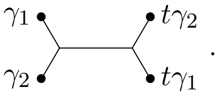


Application of $\overline{\mathrm{Hol}}$ gives:

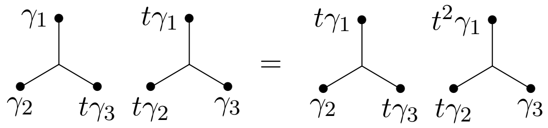


which, developed with Corollary 3.8, gives:

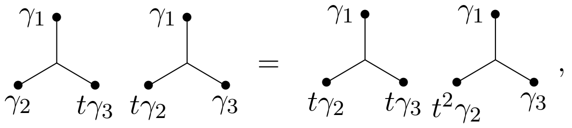
$$G_1 + aG_2 + G_4 = r$$



By Aut_t^1 and Aut_t^2 respectively, we get:

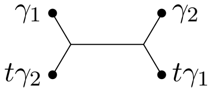


and



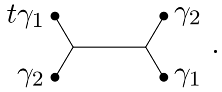
which, using Corollary 3.8, provides respectively:

$$aG_3 + 2G_4 = r$$



and

$$(a+1)G_2 + G_3 = r$$



One can check that the other relations Aut_t and $\overline{\mathrm{Hol}}$ applied to the different given forms of the G_i's do not provide further relations. □

Corollary 4.3. *The space* $\mathcal{A}_2\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)$ *admits the presentation with:*

- *as generators: the diagram* G_1 *given in Figure 7 and* $4_{\leq}$*–legs diagrams;*
- *as relations:* AS, IHX, LE, Hol, OR, LV, LD, EV *and* Aut *on* $4_{\leq}$*–legs generators and the following relation between* G_1 *and the H–diagrams given in Figure 10:*

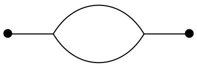
$$(1-a)(a+2)^2G_1 = 4H_3 + 2aH_2 - 2H_4 - a(a+3)H_1. \qquad (R_6)$$

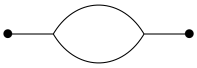
Now, we turn our attention to 4–legs generators.

Lemma 4.4. *The space* $\widehat{\mathcal{A}}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)$ *admits the presentation with:*

- *as generators: the H–diagrams* H_1*,* H_2*,* H_3*,* H_4 *given in Figure 10 and* $2_{\leq}$*–legs diagrams;*

- *as relations:* AS*,* IHX*,* LE*,* Hol*,* OR*,* LV*,* LD*,* EV *and* Aut *on* $2_{\leq}$*–legs generators and the following two relations:*

$aH_1 + 2H_2 \quad = \quad -r \quad \gamma_1$  γ_1

$aH_2 + H_3 + H_4 \quad = \quad -r \quad \gamma_1$  $t\gamma_1$.

Proof. Thanks to Lemmas 3.12 and 3.13, we only have to check that Aut_t applied to the diagrams of Figure 10 provides exactly the above two relations. This is straightforward. ☐

Corollary 4.5. *The space* $\widehat{\mathcal{A}}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)$ *admits the presentation with:*

- *as generators: the H–diagrams* H_1 *and* H_3 *given in Figure 10 and* $2_{\leq}$*–legs diagrams;*
- *as relations:* AS*,* IHX*,* LE*,* Hol*,* OR*,* LV*,* LD*,* EV *and* Aut *on* $2_{\leq}$*–legs generators.*

Proof of Proposition 4.1. Thanks to Corollaries 4.3 and 4.5, $\mathcal{A}_2\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)$ has a presentation given by the generators G_1, H_1, H_3 and all $2_{\leq}$–legs diagrams, and the relation (R_6) and all usual relations on $2_{\leq}$–legs diagrams. Using (R_6) to write H_3 in terms of the other generators, we obtain a presentation with, as generators, G_1, H_1 and $2_{\leq}$–legs diagrams and, as relations, the usual relations on $2_{\leq}$–legs diagrams. This concludes the first two points of the proposition. The third point is given by Corollary 4.5.

If $a \neq 1$, in the presentation of $\mathcal{A}_2\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)$ given in Corollary 4.3, one can remove the generator G_1 and the relation (R_6). This implies the fourth point of the proposition.

If $a = 1$, in the presentation of $\mathcal{A}_2\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)$ given in Corollary 4.3, G_1 is not subject to any relation. On the other hand, compared with Lemma 4.4, (R_6) provides then a third relation between the H_i's which holds in $\mathcal{A}_2\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)$ but not in $\widehat{\mathcal{A}}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)$. This new relation can be used to show that H_1 and H_3 are equal up to diagrams with fewer legs. This concludes the fifth point of the proposition. ☐

4.2 On the maps ι_2

The main goal of this section is to determine the injectivity and surjectivity status of the maps $\iota_2^1 : \mathcal{A}_2(\mathfrak{A},\mathfrak{b}) \to \mathcal{A}_2\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right)$ and $\iota_2^2 : \mathcal{A}_2\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big) \to \mathcal{A}_2\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right)$ when $\mathfrak{A}$ is of $\mathbb{Q}$–dimension two and cyclic. It is a direct consequence of Corollary 3.5 and Proposition 4.1 that:

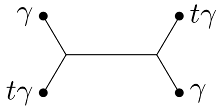
Proposition 4.6. *If* $(\mathfrak{A},\mathfrak{b})$ *is a cyclic Blanchfield module of* $\mathbb{Q}$*–dimension* 2 *with annihilator different from* $t + 1 + t^{-1}$*, then the maps* ι_2^1 *and* ι_2^2 *are injective.*

It remains to deal with injectivity when $\delta = t + 1 + t^{-1}$ and to determine the surjectivity status of the maps ι_2. We start with ι_2^1.

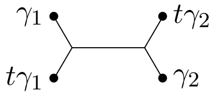
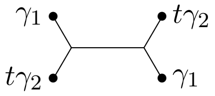
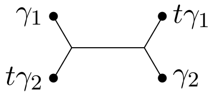
Proposition 4.7. *Let* $(\mathfrak{A},\mathfrak{b})$ *be a cyclic Blanchfield module of* $\mathbb{Q}$*-dimension two. Then the map* ι_2^1 *is injective but not surjective.*

Proof. Thanks to the first point of Proposition 4.1 and Corollary 3.3, the map ι_2^1 induces an isomorphism from $\mathcal{A}_2^{(2)}(\mathfrak{A},\mathfrak{b})$ to $\mathcal{A}_2^{(2)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)$. Hence we can work with the map $\bar{\iota}_2^1$ induced by ι_2^1 on the quotients $\mathcal{A}_2/\mathcal{A}_2^{(2)}$.

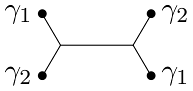
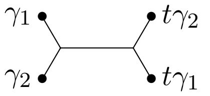
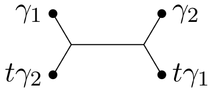
It is easy to check that $\mathcal{A}_2(\mathfrak{A},\mathfrak{b})/\mathcal{A}_2^{(2)}(\mathfrak{A},\mathfrak{b})$ is generated by the following H–diagram:

$G =$  .

By [Mou17, Proposition 7.10], $\bar{\iota}_2^1(G)$ is half the sum of all diagrams obtained from G by replacing two γ's by γ_1 and the other two by γ_2. Thanks to Aut_ξ, this gives:

$\bar{\iota}_2^1(G) =$  $+$  $+$  .

Applying an IHX relation to the first diagram, Aut_t^2 to the second one and various AS relations, it can be reformulated into:

$\bar{\iota}_2^1(G) =$  $+$  $-\,2$  .

Using Relation (R_6) and the relations of Lemma 4.4, we finally obtain:

$$\bar{\iota}_2^1(G) = \frac{1}{2}(1-a)(a+2)^2 G_1 + \frac{1}{2}(a+1)(a+2)H_1,$$

up to $2_{\leq}$–legs diagrams. It follows by the second point of Proposition 4.1 that $\bar{\iota}_2^1$ is injective but not surjective. □

We now deal with the map ι_2^2. For that, we have to study the structure of $\mathcal{A}_2\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)$. The next lemma describes the elements of $\mathrm{Aut}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)$ for a cyclic Blanchfield module $(\mathfrak{A},\mathfrak{b})$ with irreducible annihilator. For $P \in \mathbb{Q}[t^{\pm 1}]$, set $\bar{P}(t) = P(t^{-1})$.

Lemma 4.8. *If δ is irreducible in $\mathbb{Q}[t^{\pm 1}]$, then the group $\mathrm{Aut}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)$ is generated by the automorphisms*

$$\chi_P : \left\{ \begin{array}{ccc} \gamma_1 & \mapsto & P\gamma_1 \\ \gamma_2 & \mapsto & \gamma_2 \end{array} \right.$$

for $P \in \mathbb{Q}[t^{\pm 1}]$ such that $P\bar{P} = 1$ mod δ and

$$\lambda_{P,Q} : \left\{ \begin{array}{ccc} \gamma_1 & \mapsto & P\gamma_1 + Q\gamma_2 \\ \gamma_2 & \mapsto & \bar{Q}\gamma_1 - \bar{P}\gamma_2 \end{array} \right.$$

for $P, Q \in \mathbb{Q}[t^{\pm 1}]$ such that $P\bar{P} + Q\bar{Q} = 1$ mod δ.

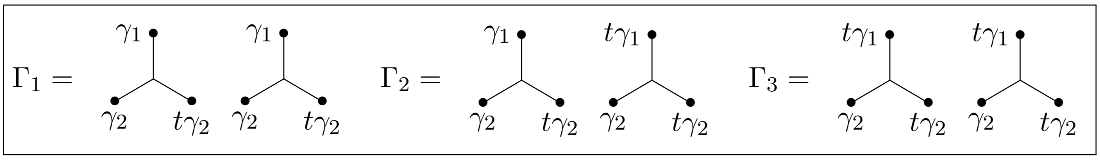


Figure 11: Some admissible YY–diagrams

Proof. In the whole proof, polynomials are considered in $\mathbb{Q}[t^{\pm1}]/_{(\delta)}$. For $P \in \mathbb{Q}[t^{\pm1}]$ such that $P\bar{P} = 1$, define

$$\chi'_P : \left\{ \begin{array}{lcl} \gamma_1 & \mapsto & \gamma_1 \\ \gamma_2 & \mapsto & P\gamma_2 \end{array} \right. ,$$

and note that $\chi'_P = \lambda_{0,1} \circ \chi_P \circ \lambda_{0,1}$. Let $\zeta \in \mathrm{Aut}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)$ and write

$$\zeta : \left\{ \begin{array}{lcl} \gamma_1 & \mapsto & P\gamma_1 + Q\gamma_2 \\ \gamma_2 & \mapsto & R\gamma_1 + S\gamma_2 \end{array} \right. .$$

Since ζ must preserve $\mathfrak{b}$, we have $P\bar{P} + Q\bar{Q} = 1$, $R\bar{R} + S\bar{S} = 1$ and $P\bar{R} + Q\bar{S} = 0$. If $Q = 0$, then $P\bar{R} = 0$, so that $R = 0$ and $\zeta = \chi_P \circ \chi'_S$. If $Q \neq 0$, then $S = -\bar{Q}^{-1}\bar{P}R$, so that

$$1 = R\bar{R} + S\bar{S} = R\bar{R}(Q\bar{Q})^{-1}(Q\bar{Q} + P\bar{P}) = R\bar{R}(Q\bar{Q})^{-1}.$$

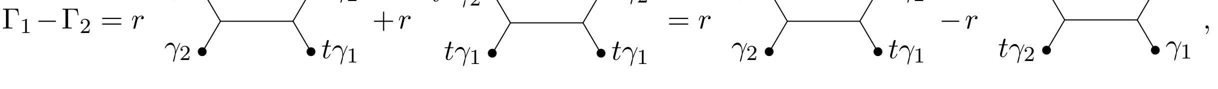
Finally $\bar{Q}^{-1}R\overline{\bar{Q}^{-1}R} = 1$ and $\zeta = \lambda_{P,Q} \circ \chi'_{\bar{Q}^{-1}R}$. □

We denote by Aut_χ and Aut_λ the subfamilies of Aut relations obtained by the action of the automorphisms χ_P and $\lambda_{P,Q}$ respectively.

Proposition 4.9. *If $(\mathfrak{A},\mathfrak{b})$ is a cyclic Blanchfield module of $\mathbb{Q}$–dimension 2, then:*

1. $\mathcal{A}_2\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big) = \mathcal{A}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)$,
2. $\mathcal{A}_2^{(2)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big) = \widehat{\mathcal{A}}_2^{(2)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)$,
3. $\mathcal{A}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big) \cong \widehat{\mathcal{A}}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)$,
4. $\mathcal{A}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)/\mathcal{A}_2^{(2)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big) \cong \widehat{\mathcal{A}}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)/\widehat{\mathcal{A}}_2^{(2)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)$*; in particular, this quotient has $\mathbb{Q}$–dimension 2.*

Proof. It is easy to see that $\mathcal{A}_2\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)/\mathcal{A}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)$ is generated by the diagrams Γ_1 and Γ_2 of Figure 11. Application of an $\overline{\mathrm{Hol}}$ relation to Γ_1 followed by a use of Corollary 3.8 gives:



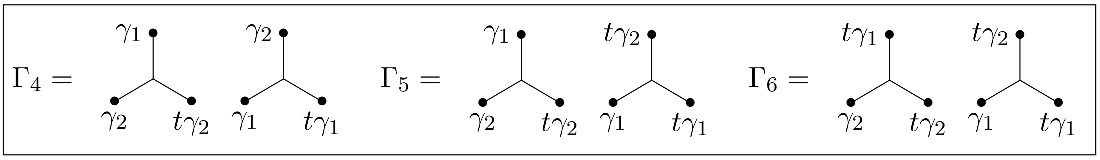
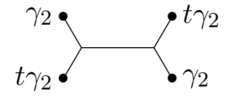


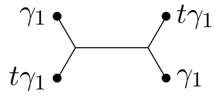
Figure 12: Some trivial admissible YY–diagrams

where the second equality comes from a $\overline{\mathrm{Hol}}$ and an AS relations on the second diagram. Application of Aut_t^1 to Γ_2 followed by a use of Corollary 3.8 and again of an Aut_t^1 relation gives:

$a\Gamma_1 + 2\Gamma_2 = r$



$= r$



,

where the second equality comes from an Aut_ξ relation. Since $a \neq -2$, it follows that both Γ_1 and Γ_2 can be expressed in term of 4–legs generators. Hence $\mathcal{A}_2\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big) = \mathcal{A}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)$, that is the first point of the proposition.

The second point follows from Proposition 4.1 (1) and Corollaries 3.2 and 3.3. Note that we have:

$$\mathcal{A}_2^{(2)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big) \cong \widehat{\mathcal{A}}_2^{(2)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big) \cong \widehat{\mathcal{A}}_2^{(2)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big) \cong \mathcal{A}_2^{(2)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big).$$

Hence, to prove the third point, we can work on the quotients $\mathcal{A}_2/\mathcal{A}_2^{(2)}$ and $\widehat{\mathcal{A}}_2/\widehat{\mathcal{A}}_2^{(2)}$.

If $a \neq 1$, the third point is given by Corollary 3.5 thanks to the first and fourth points of Proposition 4.1. Assume $a = 1$. The diagrams Γ_i for $i = 1, \ldots, 6$ represented in Figures 11 and 12 form a minimal essential set $\mathcal{E}$ of admissible YY–diagrams. Thanks to Lemmas 3.11, 3.13 and 4.8, we only need to consider $\overline{\mathrm{Hol}}^{\mathcal{E}}$, $\mathrm{Aut}_\chi^{\mathcal{E}}$ and $\mathrm{Aut}_\lambda^{\mathcal{E}}$. The $\overline{\mathrm{Hol}}$ and Aut_χ relations applied to Γ_i with $i > 3$ obvioulsy give trivial relations; check that the relations Aut_λ applied to these diagrams also give trivial relations thanks to cancellations in the decomposition.

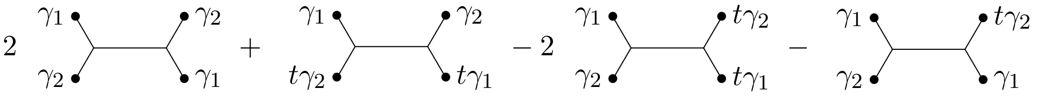
The $\overline{\mathrm{Hol}}$ relation applied to Γ_1 or Γ_2 recovers the above two relations. Up to these two relations, $\overline{\mathrm{Hol}}$ applied to Γ_3 gives a trivial relation up to $2_\leq$–legs diagrams.

It remains to write the $\mathrm{Aut}^{\mathcal{E}}$ relations corresponding to the Γ_i's with $i \leq 3$. A relation Aut_χ with an automorphism χ_P applied to Γ_3 is recovered from the relation Aut_χ with χ_{tP} applied to Γ_1. The relations Aut_χ applied to Γ_1 and Γ_2 can be written by hand. However, the relations Aut_λ imply wild computations which required the help of a computer. The program given in Appendix A checks that a relation Aut_λ applied on Γ_i for $i = 1, 2, 3$ can be recovered from the above two relations and usual relations on $4_\leq$–legs generators. This concludes the third point of the proposition.

We have seen that $\mathcal{A}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)/\mathcal{A}_2^{(2)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big) \cong \widehat{\mathcal{A}}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)/\widehat{\mathcal{A}}_2^{(2)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)$. By Corollary 3.2, we have $\widehat{\mathcal{A}}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)/\widehat{\mathcal{A}}_2^{(2)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big) \cong \widehat{\mathcal{A}}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)/\widehat{\mathcal{A}}_2^{(2)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)$. This gives the isomorphism of the fourth point. The dimension of the quotient is given by the third point of Proposition 4.1. □

Proposition 4.10. *Let $(\mathfrak{A},\mathfrak{b})$ be a cyclic Blanchfield module of $\mathbb{Q}$–dimension two, with annihilator δ. Then the map $\iota_2^2 : \mathcal{A}_2\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big) \to \mathcal{A}_2\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right)$:*

- *is an isomorphism if $\delta \neq t+1+t^{-1}$;*
- *has a non trivial kernel generated by the combination of H–diagrams*

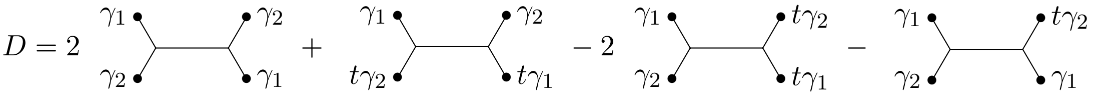


if $\delta = t+1+t^{-1}$.

Proof. First assume $\delta \neq t+1+t^{-1}$. The fourth point of Proposition 4.1 and Corollary 3.3 imply that ι_2^2 induces an isomorphism from $\mathcal{A}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)$ to $\mathcal{A}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)$. This proves the first point since $\mathcal{A}_2\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big) = \mathcal{A}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)$ by Proposition 4.9 and $\mathcal{A}_2\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big) = \mathcal{A}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)$ by the fourth point of Proposition 4.1.

Now assume that $\delta = t+1+t^{-1}$. The second point of Proposition 4.9 asserts that $\mathcal{A}_2^{(2)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big) \cong \widehat{\mathcal{A}}_2^{(2)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)$. Moreover, $\mathcal{A}_2^{(2)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big) \cong \widehat{\mathcal{A}}_2^{(2)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)$ by the first point of Proposition 4.1. Hence it follows from Corollary 3.2 that ι_2^2 is an isomorphism at the $\mathcal{A}_2^{(2)}$–level. By the first point of Proposition 4.9, the quotient $\mathcal{A}_2\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)/\mathcal{A}_2^{(2)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)$ is equal to $\mathcal{A}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)/\mathcal{A}_2^{(2)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)$, so its image by ι_2^2 is included in $\mathcal{A}_2^{(4)}\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right)/\mathcal{A}_2^{(2)}\left((\mathfrak{A},\mathfrak{b})^{\oplus 3}\right)$. Now, the H–diagram H_1 of Figure 10 is clearly in the image of ι_2^2. Finally, by Proposition 4.1 (5.i.) and Proposition 4.9 (4), the kernel of ι_2^2 has dimension 1.

More precisely, thanks to Relation (R_6), the image through ι_2^2 of



is zero. In the quotient $\widehat{\mathcal{A}}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)/\widehat{\mathcal{A}}_2^{(2)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)$, D is equal to $3(H_1 - H_3)$, which is non zero by Proposition 4.1 (3). Moreover, $\widehat{\mathcal{A}}_2^{(4)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big)/\widehat{\mathcal{A}}_2^{(2)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 3}\big) \cong \mathcal{A}_2\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)/\mathcal{A}_2^{(2)}\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)$ by Proposition 4.9 (1,4). It follows that D is non trivial in $\mathcal{A}_2\big((\mathfrak{A},\mathfrak{b})^{\oplus 2}\big)$. □

5 Case when $\mathfrak{A}$ is of $\mathbb{Q}$–dimension two and non cyclic

In this section, we assume that $(\mathfrak{A},\mathfrak{b})$ is a non cyclic Blanchfield module of $\mathbb{Q}$–dimension two. As mentioned at the beginning of Section 3, it implies that $\mathfrak{A}$ is the direct sum of two $\mathbb{Q}[t^{\pm 1}]$-modules of order $t+1$. Hence we can write:

$$\mathfrak{A} = \frac{\mathbb{Q}[t^{\pm 1}]}{(t+1)}\gamma \oplus \frac{\mathbb{Q}[t^{\pm 1}]}{(t+1)}\eta.$$

Moreover, it follows from $\mathfrak{b}$ being hermitian and non-degenerate that, up to rescaling η, $\mathfrak{b}(\gamma,\gamma) = \mathfrak{b}(\eta,\eta) = 0$ and $\mathfrak{b}(\gamma,\eta) = \frac{1}{t+1}$. Throughout the section, we consider $\{\gamma,\eta\}$ as the basis ω for $\mathfrak{A}$

and we set $f(\gamma,\gamma)=f(\eta,\eta)=0$, $f(\gamma,\eta)=\frac{1}{t+1}$ and $f(\eta,\gamma)=\frac{t}{t+1}$. Accordingly, set $\gamma_i=\xi_i(\gamma)$ and $\eta_i=\xi_i(\eta)$, for $i=1,2,3$.

Lemma 5.1. *The automorphism group* $Aut(\mathfrak{A},\mathfrak{b})$ *is generated by the following automorphisms:*

$$\mu_x:\left\{\begin{array}{lll}\gamma & \mapsto & x\gamma\\ \eta & \mapsto & x^{-1}\eta\end{array}\right. \qquad \nu:\left\{\begin{array}{lll}\gamma & \mapsto & \eta\\ \eta & \mapsto & -\gamma\end{array}\right. \qquad \rho_y:\left\{\begin{array}{lll}\gamma & \mapsto & \gamma+y\eta\\ \eta & \mapsto & \eta\end{array}\right.$$

where x runs over $\mathbb{Q}\setminus\{0,\pm1\}$ and y over $\mathbb{Q}\setminus\{0\}$.

Proof. Any automorphism ζ of $(\mathfrak{A},\mathfrak{b})$ is given by

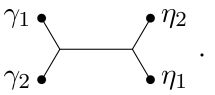
$$\zeta:\left\{\begin{array}{lll}\gamma & \mapsto & x\gamma+y\eta\\ \eta & \mapsto & z\gamma+w\eta\end{array}\right.$$

with x,y,z,w in $\mathbb{Q}$. Since ζ preserves the Blanchfield pairing $\mathfrak{b}$, we have $xw-yz=1$. If $z=0$, then $xw=1$ and $\zeta=\rho_{yx^{-1}}\circ\mu_x$. If $w=0$, then $yz=-1$ and $\zeta=\nu\circ\rho_{-xy^{-1}}\circ\mu_y$. Finally, if $zw\neq0$, then $\zeta=\mu_{w^{-1}}\circ\nu\circ\rho_{-zw}\circ\nu^{-1}\circ\rho_{yw^{-1}}$. □

We denote by Aut_μ, Aut_ν and Aut_ρ the subfamilies of Aut relations obtained by the action of the automorphisms given by μ_x, ν and ρ_y respectively on one copy of $\mathfrak{A}$ and identity on the others.

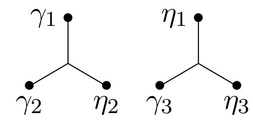
Proposition 5.2. *If $(\mathfrak{A},\mathfrak{b})$ is a non cyclic Blanchfield module of $\mathbb{Q}$–dimension two, then:*

1. $\mathcal{A}_2^{(2)}\left((\mathfrak{A},\mathfrak{b})^{\oplus3}\right)\cong\widehat{\mathcal{A}}_2^{(2)}\left((\mathfrak{A},\mathfrak{b})^{\oplus3}\right)$*;*
2. $\mathcal{A}_2\left((\mathfrak{A},\mathfrak{b})^{\oplus3}\right)=\mathcal{A}_2^{(4)}\left((\mathfrak{A},\mathfrak{b})^{\oplus3}\right)\cong\widehat{\mathcal{A}}_2^{(4)}\left((\mathfrak{A},\mathfrak{b})^{\oplus3}\right)$*;*
3. $\mathcal{A}_2\left((\mathfrak{A},\mathfrak{b})^{\oplus3}\right)/\mathcal{A}_2^{(2)}\left((\mathfrak{A},\mathfrak{b})^{\oplus3}\right)$ *is freely generated by the admissible H–diagram*

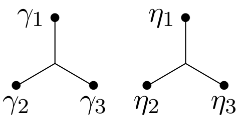


Proof. We start with the presentation given by Lemma 3.11 to deal with 6–legs generators. Let D be an admissible YY–diagram with two legs v and w labelled by the same γ_i or the same η_i. Application of any Aut_μ relation shows that the diagram D is trivial. Application of an Aut_ν, Aut_ξ or $\overline{\mathrm{Hol}}$ relation to D gives a trivial relation in Aut_ν^ω, Aut_ξ^ω or $\overline{\mathrm{Hol}}^\omega$. Application of an Aut_ρ relation to D gives in Aut_ρ^ω the relation of Aut_ν^ω obtained by applying Aut_ν to the diagram D' obtained from D by changing the labels of v and w to γ_i and η_i respectively and the linking f_{vw} to $\frac{1}{t+1}$. Hence we can remove from the generators the admissible YY–diagrams with a common label on two distinct legs without adding any relation. Then, using Lemma 3.13, it is easily seen that one can restrict the 6–legs generators to the admissible YY–diagrams:

$Y_1=$

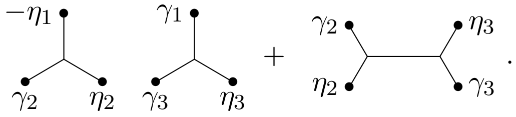


and $Y_2=$



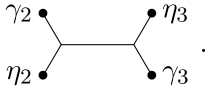
On these generators, Aut_μ and Aut_ξ act trivially, so we are left with checking the relations coming from $\overline{\mathrm{Hol}}$ and Aut_ν relations. Note however that applying these relations may change the prescribed the rational fractions on pairs of vertices, so that use of an LD relation may be needed to correct them. For instance, application of Aut_ν, regarding $\mathfrak{A}_1$, on Y_1 gives

$Y_1 =$



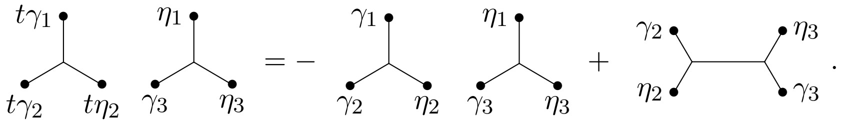
The prescribed rational fraction between the top vertices of Y_1 is indeed $\frac{1}{1+t}$, whereas the one of the 6–legs term on the right is $f(-\eta,\gamma) = \frac{-t}{1+t} = \frac{1}{1+t} - 1$; use of an LD relation is hence needed and produces the 4–legs term. Then applications of LV and Aut_ξ relations lead to

$2Y_1 =$



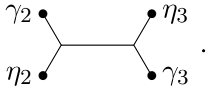
Similarly, application of $\overline{\mathrm{Hol}}$ to Y_1 gives

$Y_1 =$



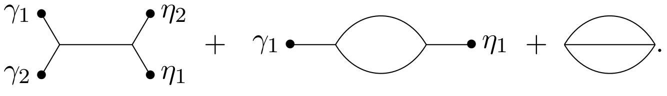
Here, the second equality is due to the fact that $tx = -x$ for any $x \in \mathfrak{A}$. The rational fraction on the top pair of vertices has to be corrected so that it corresponds to the prescribed one; this produces the 4–legs term. Once again, we get

$2Y_1 =$



Applications of Aut_ν on $\mathfrak{A}_2$ and $\mathfrak{A}_3$ give trivial relations. On Y_2, the only relations that do act non trivially are $\overline{\mathrm{Hol}}$ and Aut_ν applied simultaneously on the three $\mathfrak{A}_i$; both give:

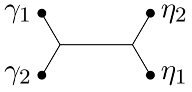
$2Y_2 = 3$



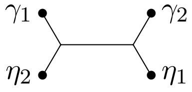
Finally, we can remove all 6–legs generators without adding any relation. This proves the second assertion.

We turn to the study of the 4–legs generators. Thanks to Lemmas 3.12 and 3.13 and removing as previously generators with a common label on two distinct legs, we are led to the diagrams:

$X_1 =$

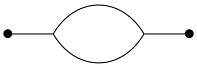


and $X_2 =$



on which we have to check the effect of the Aut_ν relations. Applying Aut_ν on $\mathfrak{A}_1$ or $\mathfrak{A}_2$ to X_1 or X_2 always gives:

$$X_1 + X_2 = - \ \gamma_1$$



$$\eta_1 \ .$$

Since no more relation arises from the 4–legs generators, this proves the first and third assertions. □

Proposition 5.3. *Let* $(\mathfrak{A}, \mathfrak{b})$ *be a non cyclic Blanchfield module of* $\mathbb{Q}$*–dimension two. Then the maps* $\iota_2^1 : \mathcal{A}_2(\mathfrak{A}, \mathfrak{b}) \to \mathcal{A}_2\left((\mathfrak{A}, \mathfrak{b})^{\oplus 3}\right)$ *and* $\iota_2^2 : \mathcal{A}_2\big((\mathfrak{A}, \mathfrak{b})^{\oplus 2}\big) \to \mathcal{A}_2\left((\mathfrak{A}, \mathfrak{b})^{\oplus 3}\right)$ *are injective. Moreover,* ι_2^2 *is surjective, while* ι_2^1 *is not.*

Proof. It is easily seen that $\mathcal{A}_2(\mathfrak{A}, \mathfrak{b})$ is generated by admissible diagrams. Such a diagram with at least four legs has necessarily two legs labelled by γ or two legs labelled by η; the relation Aut_μ implies that it is trivial. It follows that $\mathcal{A}_2(\mathfrak{A}, \mathfrak{b}) = \mathcal{A}_2^{(2)}(\mathfrak{A}, \mathfrak{b})$. Hence, by Proposition 5.2 and Corollary 3.5, ι_2^1 is injective but not surjective.

Similarly, we have $\mathcal{A}_2\big((\mathfrak{A}, \mathfrak{b})^{\oplus 2}\big) = \mathcal{A}_2^{(4)}\big((\mathfrak{A}, \mathfrak{b})^{\oplus 2}\big)$ and it follows from the second point of Proposition 5.2 and Corollary 3.3 that ι_2^2 is an isomorphism. □

A Programs

Let $(\mathfrak{A}, \mathfrak{b})$ be a cyclic Blanchfield module with annihilator $\delta = t + 1 + t^{-1}$. Let γ be a generator of $\mathfrak{A}$. As recalled at the beginning of Section 4.1, $\mathfrak{b}(\gamma, \gamma) = \frac{r}{\delta}$ *mod* $\mathbb{Q}[t^{\pm 1}]$ with $r \in \mathbb{Q}^*$. We set $\gamma_i = \xi_i(\gamma)$ for $i = 1, 2$. A $\mathbb{Q}$–basis of $\mathfrak{A}^{\oplus 2}$ is given by the $t^\varepsilon \gamma_i$ with $\varepsilon = 0, 1$ and $i = 1, 2$.

This appendix aims at determining the relations induced on $\mathcal{A}_2\big((\mathfrak{A}, \mathfrak{b})^{\oplus 2}\big)\big/\mathcal{A}_2^{(2)}\big((\mathfrak{A}, \mathfrak{b})^{\oplus 2}\big)$ by applying the Aut_λ relations to the diagrams Γ_i of Figure 11. Set

$$\lambda_{a,b,c,d} : \begin{cases} \gamma_1 & \mapsto \quad (at + b)\gamma_1 + (ct + d)\gamma_2 \\ \gamma_2 & \mapsto \quad (ct^{-1} + d)\gamma_1 - (at^{-1} + b)\gamma_2 \end{cases}$$

for $a, b, c, d \in \mathbb{Q}$ such that $a^2 + b^2 + c^2 + d^2 = 1 + ab + cd$. We wrote three programs in OCaml[1] which compute the reductions of $\lambda_{a,b,c,d}.\Gamma_1$, $\lambda_{a,b,c,d}.\Gamma_2$ and $\lambda_{a,b,c,d}.\Gamma_3$. Here, a, b, c and d are considered as parameters and all the computations are made in

$$\mathbb{Q}_{a,b,c,d} := \mathbb{Q}[a, b, c, d]\big/_{a^2 + b^2 + c^2 + d^2 - ab - cd - 1}\cdot$$

Note that every element in $\mathbb{Q}_{a,b,c,d}$ has a unique representative in $\mathbb{Q}[a, b, c, d]$ that involves no a^k with $k \geq 2$.

A.1 Implementation of the variables

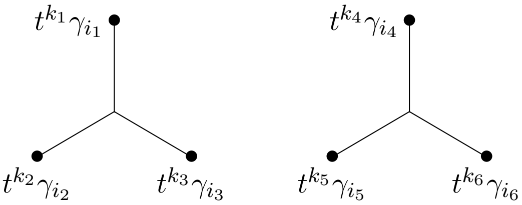
Elements of $\mathbb{Q}_{a,b,c,d}$ are implemented as lists of vectors $(\alpha, k_a, k_b, k_c, k_d) \in \mathbb{Q} \times \{0, 1\} \times \mathbb{N}^3 \subset \mathbb{Q} \times \mathbb{N}^4$, corresponding to the sum of the $\alpha a^{k_a} b^{k_b} c^{k_c} d^{k_d}$. Addition and multiplication in $\mathbb{Q}_{a,b,c,d}$

---

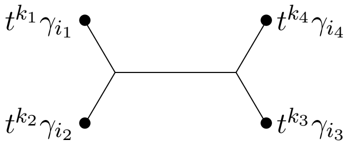
[1] avalaible at `http://www.i2m.univ-amu.fr/~audoux/Reduc_Gamma#.ml` with `#` $= 1, 2, 3$.

are implemented accordingly, using the relation $a^2 = 1 + ab + cd - b^2 - c^2 - d^2$ to remove terms with powers of a higher than 2.

Generators of $\mathcal{A}_2\big((\mathfrak{A}, \mathfrak{b})^{\oplus 2}\big)/\mathcal{A}_2^{(2)}\big((\mathfrak{A}, \mathfrak{b})^{\oplus 2}\big)$ are separated between 6–legs and 4–legs ones. The former are implemented as $\big((k_1, i_1), \dots, (k_6, i_6)\big) \in \big(\mathbb{Z} \times \{1, 2\}\big)^6$ corresponding to



and the latter as $\big((k_1, i_1), \dots, (k_4, i_4)\big) \in \big(\mathbb{Z} \times \{1, 2\}\big)^4$ corresponding to



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In both cases, the linking between legs v and w labelled by $t^{k_j}\gamma_{i_j}$ and $t^{k_\ell}\gamma_{i_\ell}$ is $f_{vw} = t^{k_j - k_\ell}\frac{r}{\delta}$. General elements of $\mathcal{A}_2\big((\mathfrak{A}, \mathfrak{b})^{\oplus 2}\big)/\mathcal{A}_2^{(2)}\big((\mathfrak{A}, \mathfrak{b})^{\oplus 2}\big)$ are implemented in two ways:

- for inputs: as linear combinations of the above generators;
- for outputs: as vectors $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \in \mathbb{Q}^6_{a,b,c,d}$ corresponding to the linear combination $\alpha_1\Gamma_1 + \alpha_2\Gamma_2 + \alpha_3 H_1 + \alpha_4 H_2 + \alpha_5 H_3 + \alpha_6 H_4$, where the H_i and the Γ_i are given in Figures 10 and 11.

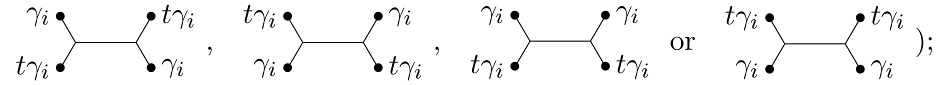
A.2 Reduction algorithms

The programs are based on two reduction algorithms `reduc4` and `reduc6`, one for 4–legs generators and one for 6–legs generators. Both algorithms take, as input, a diagram Γ implemented as an element of $\big(\mathbb{Z} \times \{1, 2\}\big)^{4 \text{ or } 6}$ representing one of the above generators and send, as output, a vector $(\alpha_1, \dots, \alpha_6) \in \mathbb{Q}^6_{a,b,c,d}$ which expresses Γ as $\Gamma = \alpha_1\Gamma_1 + \alpha_2\Gamma_2 + \alpha_3 H_1 + \alpha_4 H_2 + \alpha_5 H_3 + \alpha_6 H_4$.

The `reduc4` algorithm goes as follows.

```
Take $\big((k_1, e_1), (k_2, e_2), (k_3, e_3), (k_4, e_4)\big)$. (Call it $\Gamma$.)
Check if $e_1 + e_2 + e_3 + e_4$ is odd (that is if one of the $\mathfrak{A}_i$ appears an odd number of times),
    or if $(k_1, e_1) = (k_2, e_2)$ or $(k_3, e_3) = (k_4, e_4)$ (that is if two legs adjacent to a same
        trivalent vertex share the same label);
    if so then send $(0, 0, 0, 0, 0, 0)$.
⟶ At this point, legs sharing an adjacent trivalent vertex have distinct labels, and each $\mathfrak{A}_i$
    appears 0, 2 or 4 times in leg labels.
Check if some $k_i$ is $< 0$ or $> 1$;
```

```
    if so then send the sum of the results of reduc4 applied to the elements
        given by Corollary 3.8 to increase or decrease $k_i$.
⟶ At this point, each leg label is either some $\gamma_i$ or some $t\gamma_i$.
Check if $e_1 = e_2 = e_3 = e_4$ (that is if all legs are labelled in the same $\mathfrak{A}_i$; if so then $\Gamma$ is either
```



```
    if so then send
        $(-1)^{k_1+k_3}\Big($reduc4$\big((0,1),(1,1),(0,2),(1,2)\big)$+reduc4$\big((0,1),(1,2),(0,1),(1,2)\big)$
        +reduc4$\big((0,1),(1,2),(0,2),(1,1)\big)\Big)$ (see [Mou17, Proposition 7.10]).
⟶ At this point, each $\mathfrak{A}_i$ appears exactly twice in leg labels.
Check if $e_1 = e_2$ (that is if the two $\mathfrak{A}_1$–labelled legs are both on the left or both on the right),
    if so then send
        reduc4$\big((k_1,e_1),(k_3,e_3),(k_2,e_2),(k_4,e_4)\big)$-reduc4$\big((k_1,e_1),(k_4,e_4),(k_2,e_2),(k_3,e_3)\big)$
        (using an IHX move).
Check if $e_1 = e_4$ (that is if the two $\mathfrak{A}_1$–labelled legs are both at the top or both at the bottom),
    if so then send -reduc4$\big((k_1,e_1),(k_2,e_2),(k_4,e_4),(k_3,e_3)\big)$ (using an AS move).
⟶ At this point, each $\mathfrak{A}_i$ appears simultaneously in labels of opposite legs only.
Use $S := k_1+k_2+k_3+k_4$ and, if $S = 2$, the parity of $k_1+k_2$ and $k_1+k_2$ to determine
    to which element, among $H_1$, $H_2$, $H_3$ or $H_4$, $\Gamma$ is equal to, and send the
    corresponding output.
```

The `reduc6` algorithm goes as follows.

```
Take $\big((k_1,e_1),(k_2,e_2),(k_3,e_3),(k_4,e_4),(k_5,e_5),(k_6,e_6)\big)$. (Call it $\Gamma$.)
Check if $e_1+e_2+e_3+e_4+e_5+e_6$ is odd (that is if one of the $\mathfrak{A}_i$ appears an odd number
        of times),
    or if $(k_1,e_1) = (k_2,e_2)$ or $(k_2,e_2) = (k_3,e_3)$ or $(k_3,e_3) = (k_1,e_1)$ or $(k_4,e_4) = (k_5,e_5)$
        or $(k_5,e_5) = (k_6,e_6)$ or $(k_6,e_6) = (k_4,e_4)$ (that is if two legs adjacent to a same
        trivalent vertex share the same label);
    if so then send $(0,0,0,0,0,0)$.
⟶ At this point, legs sharing an adjacent trivalent vertex have distinct labels, and each $\mathfrak{A}_i$
    appears an even number of times in leg labels.
Check if some $k_i$ is $< 0$ or $> 1$;
    if so then send the sum of the results of reduc6 and reduc4 applied to the
        elements given by Corollary 3.8 to increase or decrease $k_i$.
⟶ At this point, each leg label is either some $\gamma_i$ or some $t\gamma_i$, and each $\mathfrak{A}_i$ appears 2 or 4 times
    in leg labels—if all legs were $\mathfrak{A}_i$–labelled, then two legs sharing a same adjacent trivalent
    vertex would have a same label.
Check if $e_1+e_2+e_3+e_4+e_5+e_6 = 8$ (that is if $\mathfrak{A}_1$ appears 4 times and $\mathfrak{A}_2$ twice in leg
        labels),
    if so then send
        reduc6$\big((k_1,3-e_1),(k_2,3-e_2),(k_3,3-e_3),(k_4,3-e_4),(k_5,3-e_5),(k_6,3-e_6)\big)$
        (using a $\mathrm{Aut}_\xi$ move).
```

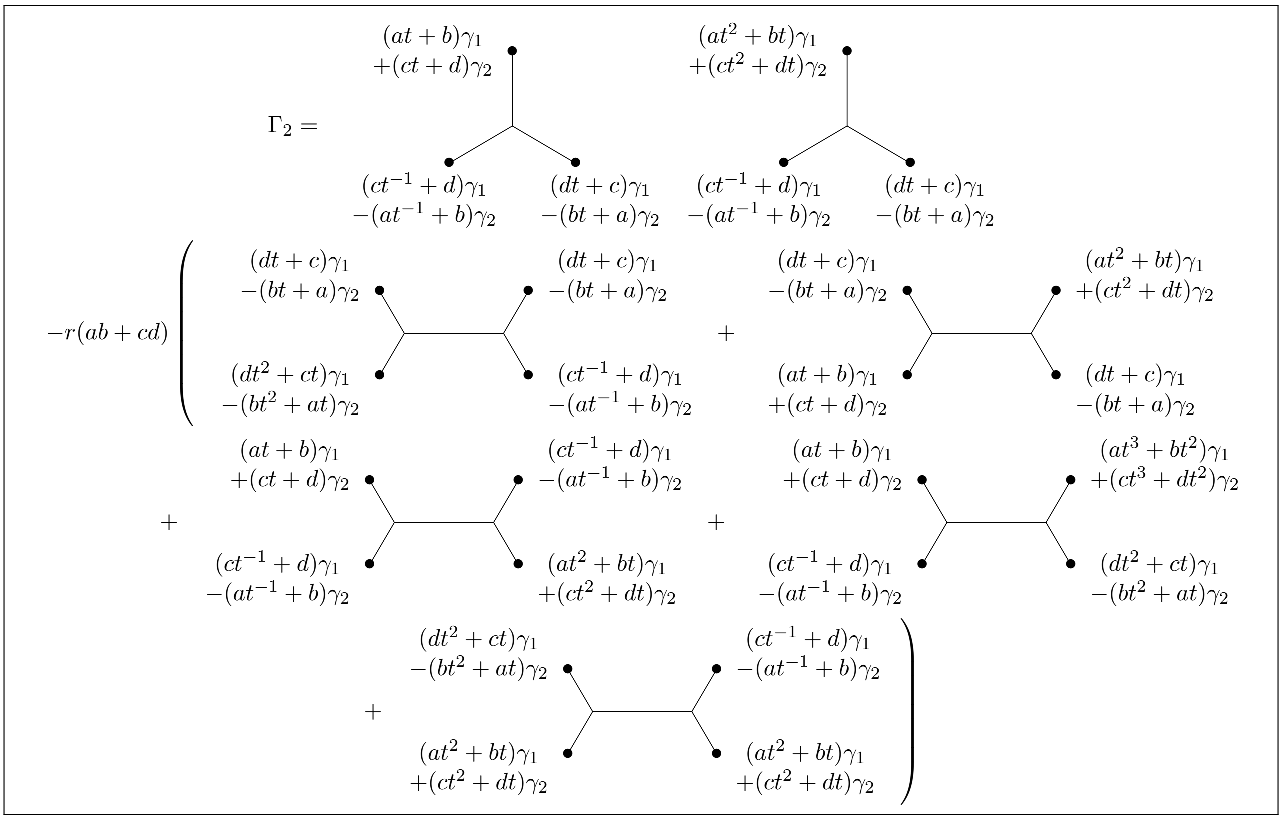


Figure 13: Input for $\lambda_{a,b,c,d}.\Gamma_2$

⟶ At this point, $\mathfrak{A}_1$ appears twice and $\mathfrak{A}_2$ four times in leg labels, and the two $\mathfrak{A}_1$–labelled legs are on distinct connected components of γ, otherwise two $\mathfrak{A}_2$–labelled legs sharing a same adjacent trivalent vertex would have a same label.

`Check if` $e_i = 1$ for $i \in \{2,3,5,6\}$ (that is if the two $\mathfrak{A}_1$–labelled legs are not both at the top), `if so then send reduc6`$\big((k'_1,e'_1),(k'_2,e'_2),(k'_3,e'_3),(k'_4,e'_4),(k'_5,e'_5),(k'_6,e'_6)\big)$ `where` $\big((k'_1,e'_1),(k'_2,e'_2),(k'_3,e'_3)\big)$ `and` $\big((k'_4,e'_4),(k'_5,e'_5),(k'_6,e'_6)\big)$ `are respectively the cyclic permutations of` $\big((k_1,e_1),(k_2,e_2),(k_3,e_3)\big)$ `and` $\big((k_4,e_4),(k_5,e_5),(k_6,e_6)\big)$ `such that` $e'_1 = e'_4 = 1$`.`

⟶ At this point, the two legs at the top are $\mathfrak{A}_1$–labelled and the four other are $\mathfrak{A}_2$–labelled, with, on each connected component of Γ, one occurence of γ_2 and one occurence of $t\gamma_2$.

`Use` $k_3 + k_5 - k_2 - k_6$ `and the parity of` $k_1 + k_4$ `to determine to which element, among` $\pm\Gamma_1$ `or` $\pm\Gamma_2$`,` Γ `is equal to, and send the corresponding output.`

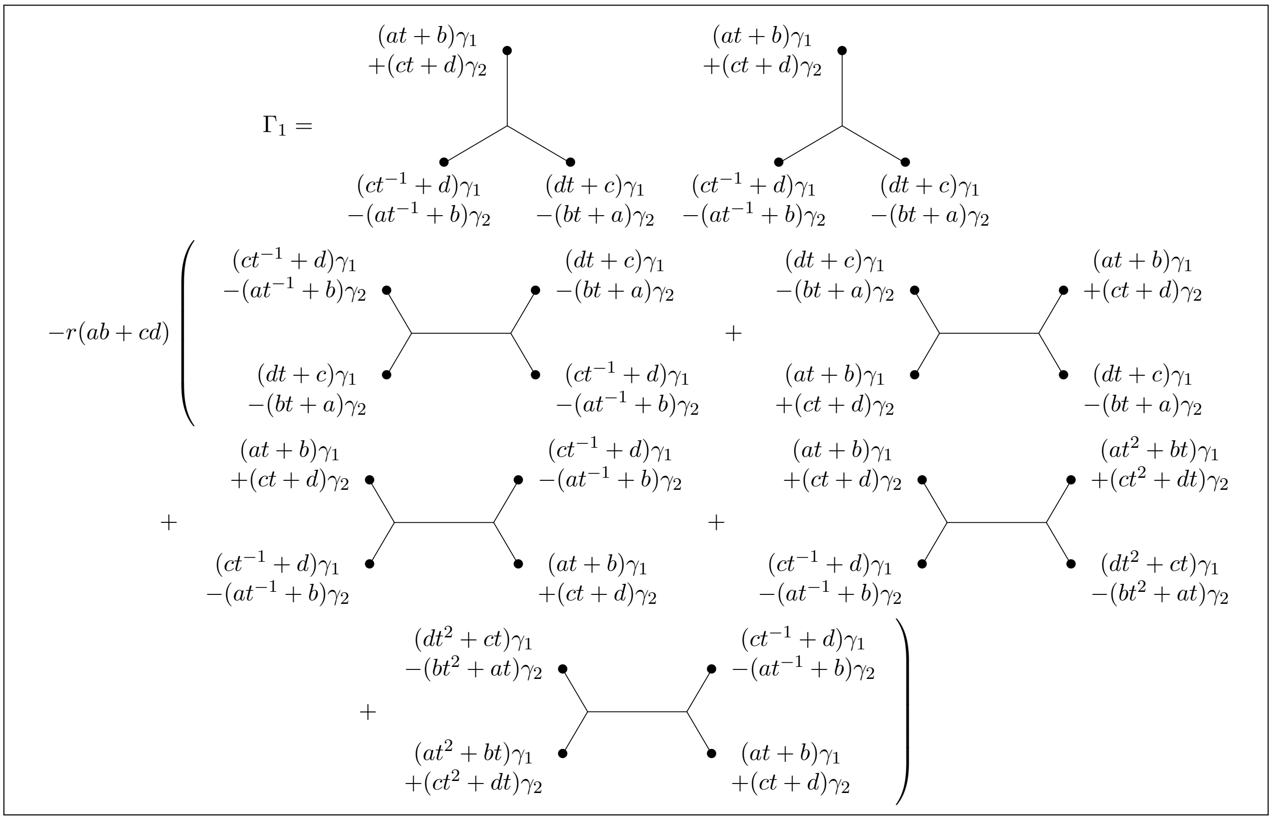
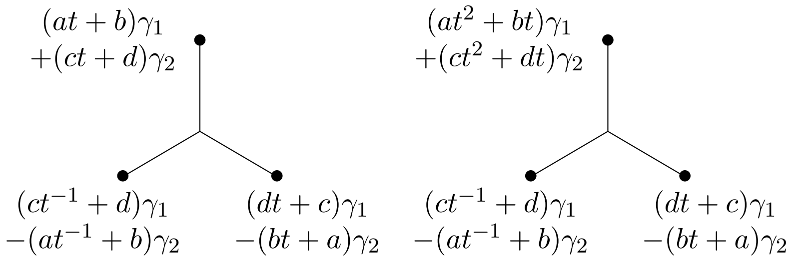


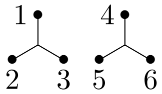
Figure 14: Input for $\lambda_{a,b,c,d}.\Gamma_1$

A.3 Computations and results

As the computation for Γ_2 is slightly more complicated than for Γ_1 and Γ_3, we start with Γ_2. The action of $\lambda_{a,b,c,d}$ on Γ_2 produces:



,

with the same linkings as in Γ_2. However, in our implementation of the diagrams as linear combinations of the generators described in Section A.1, the convention gives, for two legs v and w labelled by $P\gamma_1 + Q\gamma_2$ and $R\gamma_1 + S\gamma_2$ respectively, a linking equal to $f_{vw} = (P\bar{R} + Q\bar{S})\frac{r}{\delta}$. For instance, numbering the vertices as , we have $f_{14}^{\lambda_{a,b,c,d}.\Gamma_2} = f_{14}^{\Gamma_2} = \frac{rt^{-1}}{\delta}$ whereas the

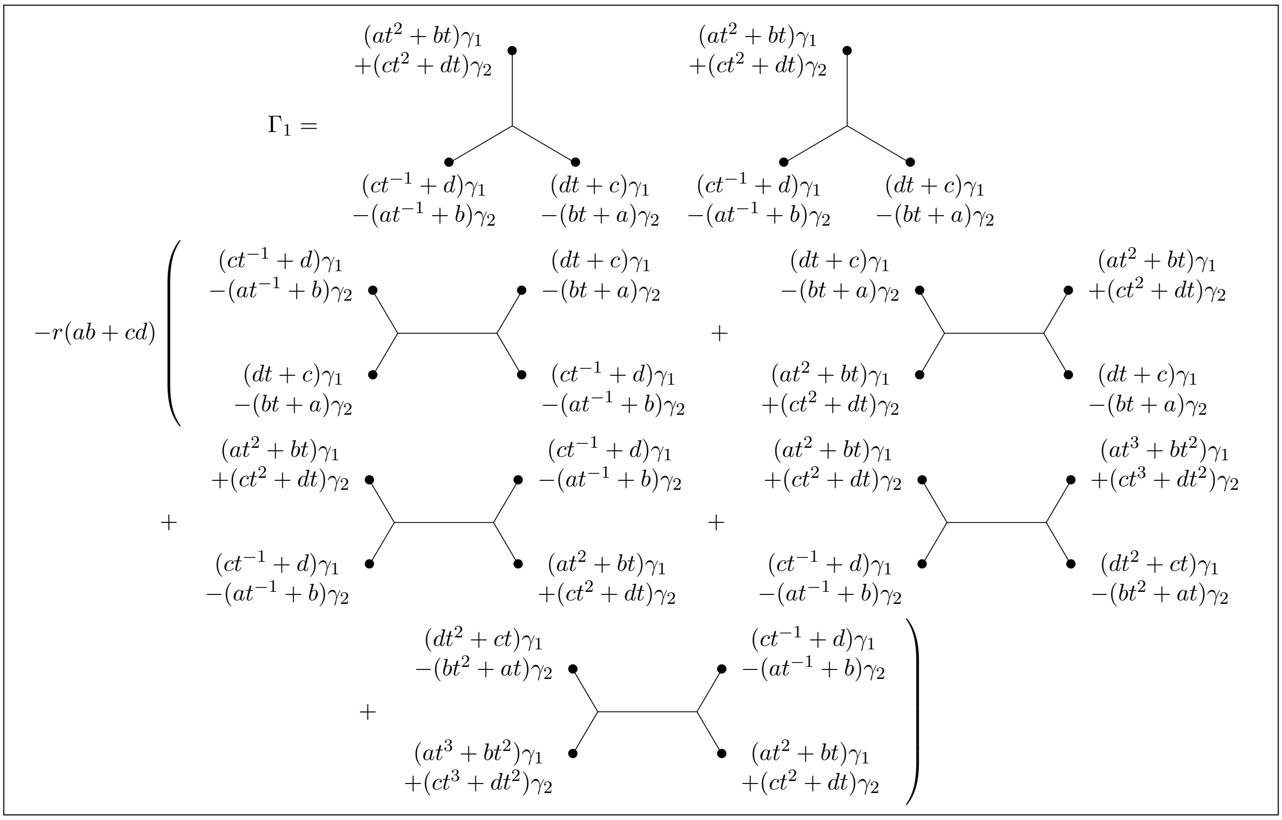
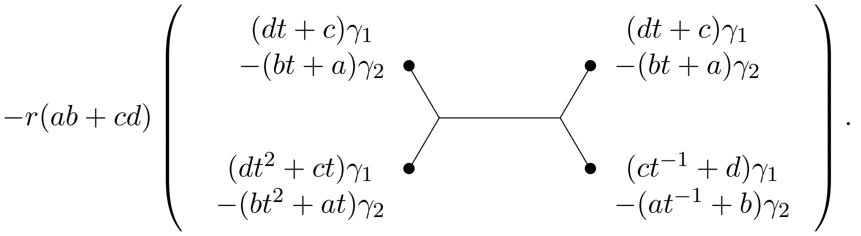


Figure 15: Input for $\lambda_{a,b,c,d}.\Gamma_3$

linking in the above diagram is

$$\begin{aligned} f_{14} &= \frac{r\Big((at+b)(at^{-2}+bt^{-1})+(ct+d)(ct^{-2}+dt^{-1})\Big)}{\delta} \\ &= \frac{r\Big((ab+cd)+(a^2+b^2+c^2+d^2)t^{-1}+(ab+cd)t^{-2}\Big)}{\delta} \\ &= \frac{r\Big((a^2+b^2+c^2+d^2-ab-cd)t^{-1}+(ab+cd)t^{-1}\delta\Big)}{\delta} \\ &= \frac{rt^{-1}}{\delta}+r(ab+cd)t^{-1}. \end{aligned}$$

This can be fixed, thanks to LV, by adding a term



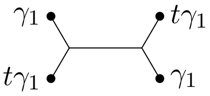
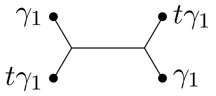
Likewise, the linking $f_{25}^{\Gamma_2}$, $f_{36}^{\Gamma_2}$, $f_{35}^{\Gamma_2}$ and $f_{26}^{\Gamma_2}$ can be fixed by adding similar 4–legs terms. All the other linkings vanish already as expected. Finally, we get the decomposition of Γ_2 given in Figure 13.

To compute the corresponding relation, we defined six matrices, one for each term in the formula of Figure 13, rows corresponding to legs and columns to each of the four monomials that appear in the leg labels. The program uses these matrices to develop with LV the six diagrams in order to get a weighted sum of generators, as they are described in Section A.1. Then, by applying either `reduc4` or `reduc6` to each term in this weighted sum, it expresses it as a linear combination of Γ_1, Γ_2 and the H_i's. Finally, the program uses the relations $H_1 = -2H_2$ and $H_4 = -H_2 - H_3$ from Lemma 4.4—which hold in $\mathcal{A}_2\big((\mathfrak{A}, \mathfrak{b})^{\oplus 2}\big)/\mathcal{A}_2^{(2)}\big((\mathfrak{A}, \mathfrak{b})^{\oplus 2}\big)$ by the same computations as in $\mathcal{A}_2\big((\mathfrak{A}, \mathfrak{b})^{\oplus 3}\big)/\mathcal{A}_2^{(2)}\big((\mathfrak{A}, \mathfrak{b})^{\oplus 3}\big)$—to reduce this linear combination in terms of Γ_1, Γ_2, H_2 and H_3 only. We end up with

$$\Gamma_2 = (b^2 + d^2 - ab - cd - 1)\Gamma_1 + (2b^2 + 2d^2 - 2ab - 2cd - 1)\Gamma_2 + r(3ab + 3cd - 3b^2 - 3d^2 + 3)H_3,$$

that is

$$(a^2 + c^2)(\Gamma_1 + 2\Gamma_2 - 3rH_3) = 0.$$

But it was already known that $\Gamma_1 + 2\Gamma_2 = r$  and the same computation as in the proof of Proposition 4.7 gives  $= H_1 + H_3 - 2H_4 = 3H_3$.

Similarly, the action of $\lambda_{a,b,c,d}$ on Γ_1 leads to the decomposition given in Figure 14. The program reduces it to

$$\Gamma_1 = (ab + cd + 1)\Gamma_1 + 2(ab + cd)\Gamma_2 - 3r(ab + cd)H_3,$$

that is

$$(ab + cd)(\Gamma_1 + 2\Gamma_2 - 3rH_3) = 0,$$

which recovers once again a previously known formula.

Finally, the action of $\lambda_{a,b,c,d}$ on Γ_3 leads to the decomposition given in Figure 15. The program reduces it to

$$\Gamma_1 = (2 - b^2 - d^2)\Gamma_1 + 2(1 - b^2 - d^2)\Gamma_2 + 3r(b^2 + d^2 - 1)H_3,$$

that is

$$(b^2 + d^2 - 1)(\Gamma_1 + 2\Gamma_2 - 3rH_3) = 0,$$

which still recovers the same previously known formula.

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