Darboux systems with a cusp point and pseudo-abelian integrals

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ABSTRACT. We study pseudo-abelian integrals associated with polynomial deformations of Darboux systems having a cuspidal singularity. Under some genericity hypothesis we provide locally uniform boundedness of on the number of their zeros.

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Key words : Pseudo-abelian integrals, First integral, Limit cycles, Darboux integrability.

1. **Introduction and main result**

The second part of Hilbert's 16th problem, asking for the maximum of the numbers of limit cycles and their relative positions for all planar polynomial differential systems of degree n. A weak version of this problem, proposed by Arnold, asking for the maximum of the numbers of isolated zeros of abelian integrals of all polynomial one-forms ω of degree n over algebraic ovals $\gamma(t) \in H_1(P^{-1}(t))$, where $P \in \mathbb{C}[x,y]$ of degree m. In [10, 8], Varchenko and Khovanskii prove the following result

Theorem 1.1. *There exist a uniform bound, depending only n and m, for the number of real zeros of abelian integrals.*

Varchenko and Khovanskii showed the existence of a uniform local bound on the number of zeros of abelian integrals when deforming the polynomial P and the polynomial form ω in their respective spaces. Next, the result is obtained using the fact that the space of parameters can be considered as being compact.

General explicit double exponential upper bound was achieved only in [2] by completely different methods. Exact upper bounds are still absent.

Arnold posed with insistence the analogous problem for more general polynomial deformations of integrable systems, in particular for deformations of system having a Darboux first integral. Then, instead of abelian integrals, one encouters pseudo-abelian integrals.

Pseudo-abelian integrals are integrals $I(t) = \int_{\gamma(t)} \frac{\eta}{M}$ of rational one-forms along cycles $\gamma(t) \subset \{H = t\}$, where

$$H = \prod_{i=1}^{k} P_i^{a_i}, \quad M = \prod_{i=1}^{k} P_i, \quad a_i \in \mathbb{R}_+^*, \quad P_i \in \mathbb{R}[x,y],$$

and M is an integrating factor. These integrals appear as principal part of the Poincaré displacement function of the deformation $\theta + \varepsilon\eta$ along $\gamma(t)$, where

$$\theta = M\frac{dH}{H}, \quad \eta = Rdx + Sdy, \quad R, S \in \mathbb{R}[x, y].$$

In [3, 9], Bobieński, Mardešić and Novikov prove the following result

Theorem 1.2. *Let H, M, η be as above. Under some genericity hypothesis, there exists a uniform bound for number of zeros of pseudo-abelian integrals associated to Darboux integrable systems close to θ.*

Here we prove an analogous result in one of non-generic cases. Another non-generic cases was studied in [1], [4] and [5].

Consider Darboux integrable system $\omega_0 = M\frac{dH}{H}$, M is an integrating factor, where

(1.1)

$$H = P_0^a \prod_{i=1}^{k} P_i^{a_i}, \quad M = P_0 \prod_{i=1}^{k} P_i, \quad P_0 = y^2 - x^3, \quad P_i \in \mathbb{R}[x, y], \quad a, a_i > 0,$$

where $P_i(0, 0) \neq 0$ for $i = 1, \ldots, k$.

Let $\omega_\varepsilon = M_\varepsilon \frac{dH_\varepsilon}{H_\varepsilon}$ be an unfolding of the form ω_0, where ω_ε are one-forms with the Darboux first integral

$$H_\varepsilon = P_\varepsilon^a \prod_{i=1}^{k} P_i^{a_i}, \quad M_\varepsilon = P_\varepsilon \prod_{i=1}^{k} P_i, \quad P_\varepsilon(x, y) = y^2 - x^3 - \varepsilon x^2. \tag{1.2}$$

Assume that the system $\omega_\varepsilon = M_\varepsilon \frac{dH_\varepsilon}{H_\varepsilon} = 0$ has a family $\{\gamma_\varepsilon(h) \subset H_\varepsilon^{-1}(h)\}$ of cycles. Consider the polynomial deformation of the system ω_ε.

$$\theta_{\varepsilon,\varepsilon_1} = \omega_\varepsilon + \varepsilon_1\eta, \quad \varepsilon_1 > 0, \quad \eta = Rdx + Sdy, \quad R, S \in \mathbb{R}[x, y]. \tag{1.3}$$

The linear part in deformation parameter ε_1 of Poincaré first return map is given by the pseudo-abelian integrals

$$I_\varepsilon(h) = \int_{\gamma_\varepsilon(h)} \frac{\eta}{M_\varepsilon}. \tag{1.4}$$

Assume that the levels curves $P_\varepsilon = 0$ and $P_i = 0$ intersect transversally and all level curves $P_i = 0, i = 1, \ldots, k$ are smooth and together with the line at infinity intersect two by two transversally and no three of them intersect in the same point.

The Darboux first integral H_ε has two critical points, one near the critical point $p_\varepsilon = (\frac{-2\varepsilon}{3}, 0)$ of P_0, which will be denoted p_e and is a center, and a saddle point $p = (0, 0)$ for $\varepsilon \neq 0$, coincide with the cusp point $p_0 = (0, 0)$ for $\varepsilon = 0$. Then, for $\varepsilon \neq 0$, two singular points p_ε and p bifurcate from p_0.

For $\varepsilon \neq 0$, we choose a compact region D which is bounded by $P_\varepsilon = 0$ and some separatrices $P_i = 0, i = 1 \cdots, k$ such that the center p_ε is outside of D. Assume that the cycles $\gamma_\varepsilon(h) \subset H_\varepsilon^{-1}(h)$ filling D, see Figure 1.

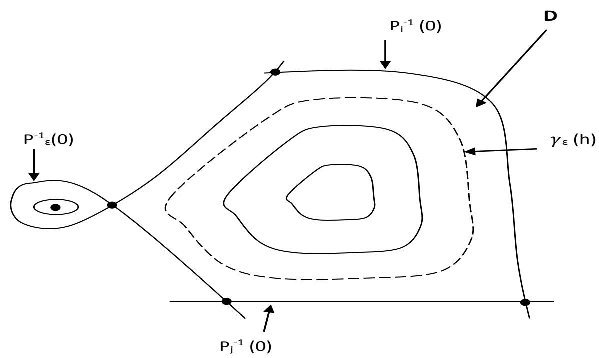

FIGURE 1. The real phase portrait of H_ε for $\varepsilon \neq 0$.

Theorem 1.3. *Let $I_\varepsilon(h)$ be the family of pseudo-abelian integrals as defined above. Then there exists an upper bound for the number of isolated zeros of the pseudo-abelian integrals $I_\varepsilon(h)$, for $h \in (0, h_0)$. The bound is locally uniform with respect to all parameters, i.e. the parameters of η, the coefficients of the polynomials P_i, the exponents a, a_i and the parameter ε.*

The Darboux systems $\omega_\varepsilon = M_\varepsilon \frac{dH_\varepsilon}{H_\varepsilon}$ have a family of cycles in the basin of the center p_ε bifurcating from $p_0 = (0, 0)$. To give a ε-uniform estimate for the number of zeros of the pseudo-abelian integrals we make the blowing-up of the cusp point of the family in the product space (x, y, ε) of phase and parameter spaces. The family blowing-ups were introduced by Denkowska and Roussarie in [6].

2. Blowing-up the cusp point

Let $\mathbb{C}^3$ be equipped with system of coordinates (x, y, ε). Denote $\mathcal{F}$ the one-dimensional foliation on $\mathbb{C}^3$ which is given by the 2-form $\omega_\varepsilon \wedge d\varepsilon, \omega_\varepsilon \in \Omega^1(\mathbb{C}^3)$. This foliation has a cuspidal singularity at the origin. We want our blow-up to simplify this singularity. This requirements leads to the quasi-homogeneous blow-up with weights $(2, 3, 2)$.

Recall the construction of the quasi-homogeneous blow-up. We define the corresponding action $\mathcal{A}$ of $\mathbb{C}^*$ on $\mathbb{C}^3 \setminus \{0\}$ by $r.(x, y, z) = (r^2x, r^3y, r^2z)$. We use this action to define weighted projective space $\mathbb{CP}^2_{2:3:2}$ as the quotient $\mathbb{C}^3 \setminus \{0\}/\mathcal{A}$. The quasi-homogeneous blow-up of $\mathbb{C}^3$ at the origin is defined as the incidence three dimensional manifold. The precise definition is the following. Let $q = (x, y, \varepsilon) \in \mathbb{C}^3$, $[(X, Y, E)] \in \mathbb{CP}^2_{2:3:2}$. Then $W = \{(p, q) \in \mathbb{CP}^2_{2:3:2} \times \mathbb{C}^3 : \exists r \in \mathbb{C} : (x, y, \varepsilon) = (r^2X, r^3Y, r^2E)\}$ which means that the point q belongs to the closure of the equivalence class defined by $p \in \mathbb{CP}^2_{2:3:2}$. The projective space $\mathbb{CP}^2_{2:3:2}$ is covered by three affine charts: $W_1 = \{x \neq 0\}$ with coordinates (Y_1, E_1), $W_2 = \{y \neq 0\}$ with coordinates (X_2, E_2) and $W_3 = \{\varepsilon \neq 0\}$ with coordinates (X_3, Y_3).

The space $\mathbb{CP}^2_{2:3:2}$ is an orbifold. More to the point, the affine chart $\Psi_2 : W_2 \to \mathbb{CP}^2_{2:3:2}$ is three-to-one: we have $\Psi_2(X_2, E_2) = \Psi_2(r^2X_2, r^2E_2) = \Psi_2(rX_2, rE_2)$ for $r = e^{\frac{2}{3}\pi i}$, so Ψ_2 branches at the point $(0, 0)$. The affine chart $\Psi_1 : W_1 \to \mathbb{CP}^2_{2:3:2}$ is two-to-one, as $\Psi_1(Y_1, E_1) = \Psi_1(-Y_1, E_1)$, and the same holds for $\Psi_3 : W_3 \to \mathbb{CP}^2_{2:3:2}$. In particular, they have branch at lines $Y_1 = 0$ and $Y_3 = 0$ correspondingly. The branching means that while the blow-up $\sigma : W \to \mathbb{C}^3$ is biholomorphism away from $\sigma^{-1}(0)$, an attempt to lift $\mathcal{F}$ to above charts on W will be complicated.

For future applications we will need explicit formula for the blow-up in the standard affine charts of W.

These affine charts define affine charts on W, with coordinates (Y_1, E_1, r_1), (X_2, E_2, r_2) and (X_3, Y_3, r_3). The blow-up σ is written as

$$\sigma_1 : \quad x = r_1^2, \qquad y = r_1^3 Y_1, \qquad \varepsilon = r_1^2 E_1 \tag{2.1}$$

$$\sigma_2 : \quad x = r_2^2 X_2, \qquad y = r_2^3, \qquad \varepsilon = r_2^2 E_2 \tag{2.2}$$

$$\sigma_3 : \quad x = r_3^2 X_3, \qquad y = r_3^3 Y_3, \qquad \varepsilon = r_3^2. \tag{2.3}$$

We apply this blow-up σ to the one-dimensional foliation $\mathcal{F}$ on $\mathbb{C}^3$ given by the intersection of $d\varepsilon = 0$ and $\omega_\varepsilon = 0$. This foliation has a cuspidal singularity at the origin. Denote by $\sigma^{-1}\mathcal{F}$ the lifting of the foliation $\mathcal{F}$ to the complement of the exceptional divisor $\sigma^{-1}(0)$. This foliation is regular outside of the preimage of $\{P_\varepsilon = 0, \varepsilon = 0\}$.

Proposition 2.1. *The foliation $\sigma^{-1}\mathcal{F}$ can be extended analytically to the exceptional divisor $\sigma^{-1}(0)$. The resulting foliation $\sigma^*\mathcal{F}$ is regular outside of the strict transform of $\{P_\varepsilon = 0, \varepsilon = 0\}$.*

Proposition 2.2. *The singularities of the resulting foliation $\sigma^*\mathcal{F}$ on $\sigma^{-1}(0)$ are located at the points $p_1 = (0, 1, 0), p_2 = (0, -1, 0), p_3 = (0, 0, -\frac{3}{2})$ and $p_c = (\frac{-2}{3}, 0, 0)$.*

Proof. The quasi-homogeneous blow-up $\sigma : W \to \mathbb{C}^3$ is a biholomorphism outside the exceptional divisor $\sigma^{-1}(0)$, all singularties of the foliation $\sigma^*\mathcal{F}$ outside $\sigma^{-1}(0)$ correspond to singularities of $\mathcal{F}$.

Let $\sigma_1^*\mathcal{F}$ be the restriction of $\sigma^*\mathcal{F}$ to the chart W_1. On the exceptional divisor, the foliation $\sigma_1^*\mathcal{F}$ has a first integral $F_1 = E^{-3}(Y^2 - E - 1) = t$. The corresponding logarithmic form is given by

$$
\begin{aligned}
E(Y^2 - E - 1)\frac{dF_1}{F_1} &= -3(Y^2 - E - 1)dE + Ed(Y^2 - E) \\
&= (-3Y^2 + 2E + 3)dE + 2EY\,dY,
\end{aligned}
$$

and the singular points of the foliation are $(r = 0, Y = \pm 1, E = 0)$ and $(0, 0, -\frac{3}{2})$. Note that the first two correspond to the same point on $\mathbb{CP}^2_{2:3:2}$. The second one corresponds to $p_c = (-\frac{2}{3}, 0, 0)$ in the chart W_3 (see Figure 2).

ptional divisor are the line of centers $(X_3 = \frac{-2}{3}, Y_3 = 0)$. ∎

Remark 2.3. Strictly speaking, the point p_1 is not a saddle of $\sigma^*\mathcal{F}$ as it is not an isolated singularity: on each leaf $\{\varepsilon = Const\}$ there is a saddle converging to p_1, so the lifting $\sigma^*\mathcal{F}$ has a whole line of singular points $(0, 0, r)$ transversal to $\sigma^{-1}(0)$ and intersecting it at p_1. However, each point $(0, 0, r)$ is a saddle on its leaf $\{\varepsilon = Const\}$.

3. Proof of Theorem 1.3

In this section we first take benefit from the blowing-up in the family performed in the previous section to prove Theorem 1.3.

Let $\mathcal{F}$ be the one-dimensional foliation on $\mathbb{C}^3$ introduced in the previous section and $\mathfrak{sing}\mathcal{F} = \{p_4, \ldots, p_k\}$ be the set of its saddles points.

3.1. **Polycycles.** Let $t := \frac{\varepsilon^3}{h}$ and $G := \frac{(\sigma^*\varepsilon)^3}{\sigma^* H(\varepsilon, x, y)}$. The phase portait of the resulting foliation $\sigma_1^*\mathcal{F}$ on exceptional divisor is given by the levels curves $\{G = t\}$. After the making of the blowing-up we obtain, on each leaf $\{\varepsilon = Const\}$, a hyperbolic polycycle (i.e. each intersection of consecutive edges we have a saddle point)

$$
\delta = \begin{cases} \delta_0 \cup \delta_4 \cup \ldots \cup \delta_k, \\ \delta_1 \cup \delta_2 \cup \delta_4 \cup \ldots \cup \delta_k, \\ \delta_3 \cup \delta_4 \cup \ldots \cup \delta_k, \end{cases} \tag{3.1}
$$

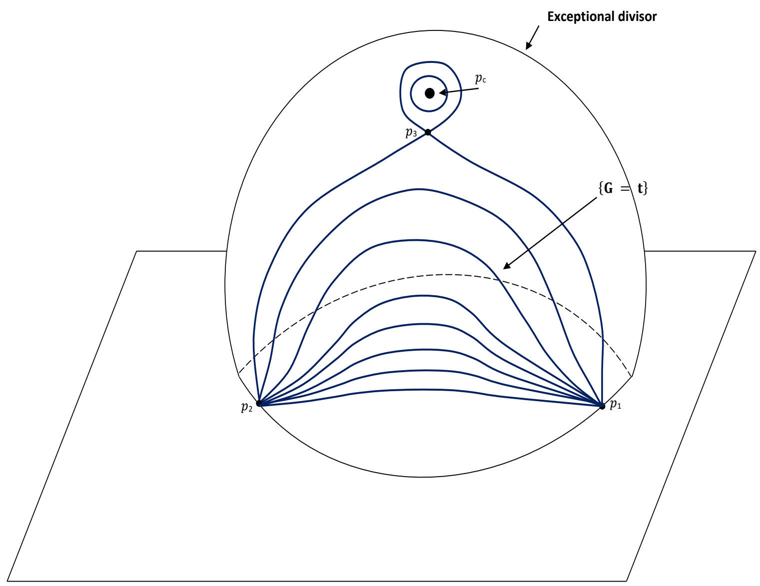

FIGURE 2. The levels curves $\{G = t\}$.

where $\delta_3 \subset \{G = t\}^{\mathbb{R}}$ and $\{\dots\}^{\mathbb{R}}$ denotes the real part of a complex analytic set, see Figure 3.

Let $\mathfrak{sing}\sigma^*\mathcal{F} = \{p_1, \dots, p_k\}$ be the set of singular points of the resulting foliation $\sigma^*\mathcal{F}$. Let us fix a polycycle δ of family (3.1) and denote $\delta(\varepsilon, h)$ its corresponding cycle, see Figure 3. We define

$$J(\varepsilon, h) = \int_{\delta(\varepsilon,h)} \sigma^* \frac{\eta}{M_\varepsilon}. \tag{3.2}$$

3.2. **Variation relations.** Define the variation operator Var as the difference between counterclockwise and clockwise continuation of F

$$Var_{(h,\beta)}F(h) = F(he^{i\beta\pi}) - F(he^{-i\beta\pi}).$$

The integral J admits analytic continuation to a universal cover of a product $\mathbb{D}_{0,\varepsilon} \times \mathbb{D}_{0,h}$ of two small punctured discs.

Proposition 3.1. *The integral $J(\varepsilon, h)$ satisfies the following iterated variations equation*

$$Var_{(h,\alpha_1)} \circ \dots \circ Var_{(h,\alpha_{k+1})} J(\varepsilon, h) = 0, \quad \alpha_i \in \mathbb{R}[a, a_1, \dots, a_k]. \tag{3.3}$$

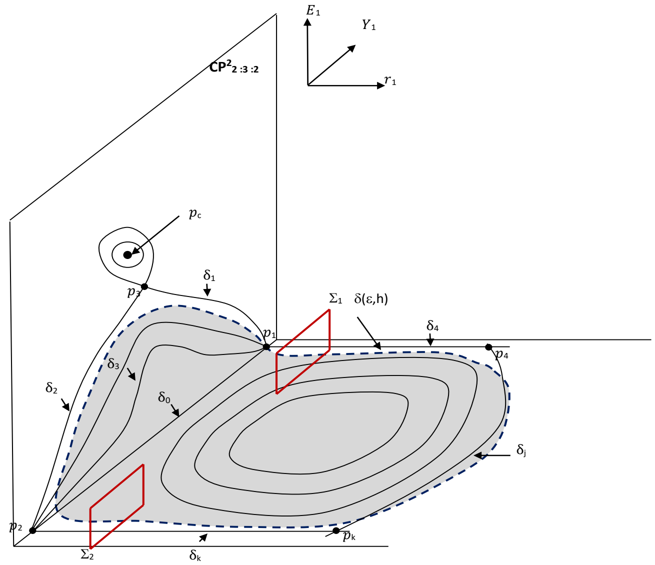

FIGURE 3. Real phase portrait of the foliation $\sigma^*\mathcal{F}$.

Proof. The globally multivalued Darboux first integral is in fact analytic in the neighborhood of the origin, so the results of [3] are applicable. Concretely, let us fix a leaf $\{\varepsilon = Const\}$ and some hyperbolic polycycle $\delta = \coprod_i \delta_i$ of family (3.1). Using a partition of unity multiplying the form $\sigma^* \frac{\eta}{M_\varepsilon}$ we can consider semilocal problem with a relative cycle close to one separatrix of the polycycle. Precisely, let δ_i be a some separatrix (connecting saddles p_j and p_k) of the polycycle δ with exponent α_i and two saddles points $p_j = \delta_i \cap \delta_j, p_k = \delta_i \cap \delta_k$, where δ_j, δ_k two separatrices of δ with exponents α_j and α_k, respectively. Let $\delta_i(\varepsilon, h)$ be its corresponding relative cycle. The respective iterated variations of the relative cycle $\delta_i(\varepsilon, h)$ is a closed loop which is either the commutator loop if $\alpha_j \neq \alpha_k$ or the figure eight loop if $\alpha_j = \alpha_k$.

Finally, using the commutativity of the variation operator Var and the univaluedness of the blown-up form $\sigma_1^* \frac{\eta}{M_\varepsilon}$, we obtain

$$Var_{(h,\alpha_1)} \circ \cdots \circ Var_{(h,\alpha_{k+1})} J(\varepsilon, h) = 0.$$

$\square$

Proposition 3.2. *The variation of the integral J with respect to ε is an integral of the blown-up form $\sigma^* \frac{\eta}{M_\varepsilon}$ along the figure eight cycle*

$$Var_{(\varepsilon,1)} J(\varepsilon, h) = \int_{eight\ figure} \sigma^* \frac{\eta}{M_\varepsilon} \tag{3.4}$$

Proof. In the chart W_1, let (Y, E, r) be a suitable coordinate system near the edge δ_0 (Figure 3) such that the blown-up foliation $\sigma_1^* \mathcal{F}$ is given by two first integrals $\varepsilon = r^2 E$ and $h = r^6 (Y-1)(Y+1)$. Near p_1 (respectively p_2), let fix a transversal $\Sigma_1 = \{r = 1\}$ (resp $\Sigma_2 = \{r = 1\}$) to the edge $\{E = Y - 1 = 0\}$ (respectively $\{E = Y + 1 = 0\}$), and introduce a new local coordinate $Z = (Y+1)(Y-1)$, so E, Z are local coordinates. The restriction of the foliation $\sigma_1^* \mathcal{F}$ on Σ_1 (respectively Σ_2) is given by two first integrals $\varepsilon = E, h = Z$. Let $(Z_1, E_1) = (h, \varepsilon) = \delta(\varepsilon, h) \cap \Sigma_1$ be the starting point and $(Z_2, E_2) = (h, \varepsilon) = \delta(\varepsilon, h) \cap \Sigma_2$ be the end point, see Figure 3. So, as ε makes a full turn around $\varepsilon = 0$ in the (ε, h)-space with h fixed, each point $(Z_i, E_i)_{i=1,2}$ makes a full turn around $E = 0$ in Σ_i with Z remains fixed, geometrically means that a eight figure produced near the edge δ_0 e.g $Var_{(\varepsilon,1)} \delta(\varepsilon, h) =$ *eight figure*. On the other hand, for a finite distance to the exceptional divisor $\{r_1 = 0\}$ and near a each edge $\delta_i, i = 4, \ldots, k$, the foliation $\sigma_1^* \mathcal{F}$ is analytic in variable E_1 and consequently we have $Var_{(\varepsilon,1)} \delta(\varepsilon, h) = 0$. Using the partition of unity, conclude that have

$$\begin{aligned} Var_{(\varepsilon,1)} J(\varepsilon, h) &= Var_{(\varepsilon,1)} \int_{\delta(\varepsilon,h)} \sigma_1^* \frac{\eta}{M_\varepsilon} = Var_{(\varepsilon,1)} \sum_j \int_{\delta_j(\varepsilon,h)} \sigma_1^* \frac{\eta}{M_\varepsilon} \\ &= \sum_j Var_{(\varepsilon,1)} \int_{\delta_j(\varepsilon,h)} \sigma_1^* \frac{\eta}{M_\varepsilon} = \int_{\text{eight loop}} \sigma_1^* \frac{\eta}{M_\varepsilon}. \end{aligned}$$

□

3.3. **Proof of Theorem 1.3.** The integral $J(\varepsilon, h)$ can be viewed as the pull-back of the pseudo-abelian integrals $I_\varepsilon(h) := I(\varepsilon, h)$ by the blowing-up map σ. The proof of Theorem 1.3 is now reduced to the proof of

Theorem 3.3. *The number $\#\{h \in (0, h_0) : J(\varepsilon, h) = 0\}$ is uniformly bounded in ε.*

Proof. Let $\mu > 0$ be sufficiently small and $\alpha_i \in \{\alpha_1, \ldots, \alpha_{k+1}\}$. In order to apply the argument principle to $J(\varepsilon, h)$, we define Γ_i a simply connected region with boundary $\partial\Gamma_i = C_R^{\alpha_i} \cup C_\mu^{\alpha_i} \cup C_\pm^{\alpha_i}$ where $C_R^{\alpha_i} = \{|h| = R, |\arg(h)| \leq \alpha_i \pi\}$ (big arc), $C_\mu^{\alpha_i} = \{|h| = \mu, |\arg(h)| \leq \alpha_i \pi\}$ (small arc) and $C_\pm^{\alpha_i} = \{\mu < |h| < R, |\arg(h)| = \pm\alpha_i \pi\}$ (segments).

The argument principle says that:

$$\begin{aligned} \#\{h \in \Gamma_i : J(\varepsilon, h) = 0\} \leq \frac{1}{2\pi} \Delta \arg_{\partial\Gamma_i} J(\varepsilon, h) &= \frac{1}{2\pi} \Delta \arg_{C_R^{\alpha_i}} J(\varepsilon, h) \\ &+ \frac{1}{2\pi} \Delta \arg_{C_\pm^{\alpha_i}} J(\varepsilon, h) + \frac{1}{2\pi} \Delta \arg_{C_\mu^{\alpha_i}} J(\varepsilon, h), \end{aligned}$$

where the circular arcs and the segments are taken with a suitable parameterization.

(1) The estimation of the increment of argument $\Delta \arg_{C_R^{\alpha_i}} J(\varepsilon, h)$ of $J(\varepsilon, h)$ along $C_R^{\alpha_i}$ is again due to Gabrielov's theorem [7].

(2) The estimation of the increment of argument $\Delta \arg_{C_\pm^{\alpha_i}} J(\varepsilon, h)$ of $J(\varepsilon, h)$ along segments $C_+^{\alpha_i}$ and $C_-^{\alpha_i}$ reduces to estimating zeros of $Var_{(h,\alpha_i)} J(\varepsilon, h)$ which is again a pseudo-abelian integral over different cycle, but now satisfying iterated variation equation of smaller lenght. As in [3, 9], by induction on this lenght, it has uniformly bounded number of zeros.

(3) As a consequence of equation (3.3) the function admits, near $h = 0$, the following asymptotic expansion

$$J(\varepsilon, h) = \sum_{i=1}^{k+1} \sum_{j=0}^{k} \phi_i(\varepsilon) h^{\alpha_i} \log^j h. \tag{3.5}$$

Existence of this asymptotic expansion implies that for sufficiently small μ the increment $\Delta \arg_{C_\mu^{\alpha_i}} J(\varepsilon, h)$ is at most $2\alpha_i M(\varepsilon) + 1$, i.e. is bounded by the rate of growth of $J(\varepsilon, h)$ as $h \to 0$. The latter can be easily bounded from above, uniformly in ε, as in [3, 9].

All the above constructions depend analytically on parameters like coefficients of the polynomials P_i, exponents a, a_i and coefficients of the form η.

$\square$

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