THE QUADRATIC LINKING DEGREE

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Abstract. By using motivic homotopy theory, we introduce an analogue in algebraic geometry of oriented links and their linking numbers. After constructing the quadratic linking degree — our analogue of the linking number which takes values in the Witt group of the ground field — and exploring some of its properties, we give a method to explicitly compute it. We illustrate this method on a family of examples which are analogues of torus links, in particular of the Hopf and Solomon links.

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1. Introduction

In 1999, Morel and Voevodsky founded motivic homotopy theory (see [MV99]) in order to import topological methods into algebraic geometry. The goal of this paper is to explore the possibility of defining an analogue of knot theory in algebraic geometry by using motivic homotopy theory. Specifically, we define analogues, over a perfect field F, of oriented links with two components. We replace the circle $\mathbb{S}^1$ with $\mathbb{A}^2_F \setminus \{0\}$ and the 3-sphere $\mathbb{S}^3$ with $\mathbb{A}^4_F \setminus \{0\}$. We then define an analogue of the linking number, which in knot theory is an invariant of oriented links with two components: the number of times one of the oriented components turns around the other oriented component. We call this analogue the quadratic linking degree.

Theorem-Definition 1.1 (Quadratic linking degree). *Let $Z = \mathbb{A}^2_F \setminus \{0\} \sqcup \mathbb{A}^2_F \setminus \{0\} \subset \mathbb{A}^4_F \setminus \{0\}$ be an oriented link with two components (see Definition 2.2). There exist two elements of the Chow-Witt group $\widetilde{CH}^1((\mathbb{A}^4_F \setminus \{0\}) \setminus Z)$ — called Seifert classes (see Definition 2.6) — such that their intersection product in $\widetilde{CH}^2((\mathbb{A}^4_F \setminus \{0\}) \setminus Z)$ and its image by the boundary map $\partial : \widetilde{CH}^2((\mathbb{A}^4_F \setminus$*

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$\{0\}) \setminus Z) \to H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\det(\mathcal{N}_{Z/\mathbb{A}^4_F\setminus\{0\}})\})$ — *called the quadratic linking class (see Definition 2.8) — only depend on the oriented link Z. Denoting by $\mathrm{W}(F)$ the Witt group of F, we call the image of the quadratic linking class by the isomorphism* $H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\det(\mathcal{N}_{Z/\mathbb{A}^4_F\setminus\{0\}})\}) \to \mathrm{W}(F) \oplus \mathrm{W}(F)$ *the quadratic linking degree (see Definition 2.11).*

Note that this definition is similar to the definition in knot theory of the linking class $(n[\omega_{K_1}], -n[\omega_{K_2}]) \in H^1(K_1 \sqcup K_2) = \mathbb{Z}[\omega_{K_1}] \oplus \mathbb{Z}[\omega_{K_2}]$ of an oriented link $K_1 \sqcup K_2$ (of volume forms $\omega_{K_1}, \omega_{K_2}$) of linking number n.

Let us illustrate this definition on the Hopf link $Z = \{x = y = 0\} \sqcup \{z = t = 0\} \subset \mathrm{Spec}(F[x, y, z, t]) \setminus \{0\}$ (see Example 4.1). Its Seifert classes are the classes of $\langle x\rangle \otimes \overline{y}^*$ and $\langle z\rangle \otimes \overline{t}^*$ in $\widetilde{CH}^1((\mathbb{A}^4_F \setminus \{0\}) \setminus Z)$, their intersection product is the class of $\langle xz\rangle \otimes (\overline{t}^* \wedge \overline{y}^*)$ in $\widetilde{CH}^2((\mathbb{A}^4_F \setminus \{0\}) \setminus Z)$ and the quadratic linking class is the class of $-\langle z\rangle\eta \otimes (\overline{t}^* \wedge \overline{x}^* \wedge \overline{y}^*) \oplus \langle x\rangle\eta \otimes (\overline{y}^* \wedge \overline{z}^* \wedge \overline{t}^*)$ in $H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\det(\mathcal{N}_{Z/\mathbb{A}^4_F\setminus\{0\}})\})$, which gives $(-1, 1) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$ as quadratic linking degree.

In Section 2, we give the definitions of the quadratic linking class and degree, then we determine how they depend on choices of orientations and of parametrizations of $\mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$ (see Lemma 2.13) and deduce invariants of the quadratic linking degree (see Corollaries 2.14 and 2.15 and Theorem 2.17). For instance, in the case $F = \mathbb{R}$, the absolute values of the components of the quadratic linking degree (which are in $\mathrm{W}(\mathbb{R}) \simeq \mathbb{Z}$) are invariant under changes of orientations and of parametrizations of $\mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$. This is similar to the fact that the absolute value of the linking number does not depend on choices of orientations. In the general case, the ranks modulo 2 of the components of the quadratic linking degree are invariants and more importantly we have the following Lemma-Definition and Theorem:

Lemma-Definition 1.2. *Let $d = \sum_{i=1}^{n} \langle a_i\rangle \in \mathrm{W}(F)$. There exists a unique sequence of abelian groups $Q_{d,k}$ and of elements $\Sigma_k(d) \in Q_{d,k}$, where k ranges over the nonnegative even integers, such that:*

- $Q_{d,0} = \mathrm{W}(F)$ *and* $\Sigma_0(d) = 1 \in Q_{d,0}$;
- *for each positive even integer k, $Q_{d,k}$ is the quotient group $Q_{d,k-2}/(\Sigma_{k-2}(d))$;*
- *for each positive even integer k,* $\Sigma_k(d) = \sum_{1\le i_1<\cdots<i_k\le n} \langle \prod_{1\le j\le k} a_{i_j}\rangle \in Q_{d,k}$.

Theorem 1.3. *Let $\mathscr{L}$ be an oriented link with two components and k be a positive even integer. We denote the quadratic linking degree of $\mathscr{L}$ by $\mathrm{Qld}_{\mathscr{L}} = (d_1, d_2) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$. Then $\Sigma_k(d_1)$ and $\Sigma_k(d_2)$ are invariant under changes of orientations and of parametrizations of $\mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$.*

In Section 3 we give a method to explicitly compute the quadratic linking class (see Theorem 3.1) and the quadratic linking degree (see Theorem 3.2) when the link $Z_1 \sqcup Z_2 \subset \mathbb{A}^4_F \setminus \{0\}$ is such that for each $i \in \{1, 2\}$ the closure $\overline{Z_i} \subset \mathbb{A}^4_F$ of Z_i is given by two irreducible equations $\{f_i = 0, g_i = 0\}$ such that $\{g_1 = 0, g_2 = 0\}$ is of codimension 2 in $\mathbb{A}^4_F \setminus \{0\}$.

In Section 4 we compute the quadratic linking class and the quadratic linking degree on several examples. The Examples 4.2 (which we call binary links) showcase the usefulness of the invariant Σ_2 by showing that it can distinguish between an infinity of different links. The Examples 4.3 are inspired by the torus links $T(2, 2n)$ of linking number n (the Hopf link if $n = 1$, the Solomon link if $n = 2$ and the n-gonal link (two intertwined n-gons) if $n \geq 3$).

In Appendix A we give an explicit definition of the residue morphisms of Milnor-Witt K-theory (see Theorem A.11). This explicit definition is used in Sections 3 and 4.

In Appendix B we recall some useful notions about the Rost-Schmid complex and its groups.

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2. The quadratic linking degree

In this section, we define oriented links with two components, oriented fundamental classes (and cycles), Seifert classes (and divisors) relative to the link, the quadratic linking class and the quadratic linking degree of the link. We then explicit how the quadratic linking class and the quadratic linking degree depend on choices of orientations and of parametrizations of $\mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$ and deduce a series of invariants of the quadratic linking degree.

2.1. **Conventions and notations.** Throughout this section, F is a perfect field, we put $\mathbb{A}^2_F = \mathrm{Spec}(F[u,v])$, $\mathbb{A}^4_F = \mathrm{Spec}(F[x,y,z,t])$ and $X := \mathbb{A}^4_F \setminus \{0\}$.

For Z a smooth closed subscheme of a smooth scheme Y, we denote by $\mathcal{N}_{Z/Y}$ the normal sheaf of Z in Y, i.e. the dual of the $\mathcal{O}_Z$-module $\mathscr{I}_Z/\mathscr{I}_Z^2$ with $\mathscr{I}_Z$ the ideal sheaf of Z in Y.

We denote the usual generators of the Milnor-Witt K-theory ring of F by $[a] \in K_1^{\mathrm{MW}}(F)$ (with $a \in F^*$) and $\eta \in K_{-1}^{\mathrm{MW}}(F)$ (see [Mor12, Definition 3.1]). We put $\langle a\rangle := 1 + \eta[a] \in K_0^{\mathrm{MW}}(F)$.

For Y a smooth F-scheme, $j \in \mathbb{Z}$ and $\mathcal{L}$ an invertible $\mathcal{O}_Y$-module, we denote the Rost-Schmid complex by $\mathcal{C}(Y, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\})$ (see Definition B.2) and the i-th Rost-Schmid group of this complex by $H^i(Y, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\})$ (see Definition B.5).

We identify $H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}_0^{\mathrm{MW}})$ with $\mathrm{W}(F)$ via the (noncanonical) isomorphism $\zeta : H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}_0^{\mathrm{MW}}) \to \mathrm{W}(F)$ which factorizes as follows:

$$H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}_0^{\mathrm{MW}}) \xrightarrow{\partial} H^0(\{0\}, \underline{K}_{-2}^{\mathrm{MW}}\{\det(\mathcal{N}_{\{0\}/\mathbb{A}^2_F})\}) \longrightarrow K_{-2}^{\mathrm{MW}}(F) \xrightarrow{\eta^2 \mapsto 1} \mathrm{W}(F)$$

where the map in the middle is induced by the isomorphism $\det(\mathcal{N}_{\{0\}/\mathbb{A}^2_F}) \to \mathcal{O}_{\{0\}} \otimes \mathcal{O}_{\{0\}}$ which sends $\overline{u}^* \wedge \overline{v}^*$ to $1 \otimes 1$ (see Definition B.15).

2.2. **Definitions of the quadratic linking class and degree.** In this subsection, we give a series of definitions which conclude with the definitions of the quadratic linking class and the quadratic linking degree of an oriented link with two components.

In order to define oriented links with two components, we need the following definition (which was given by Morel in [Mor12]).

Definition 2.1 (Orientation of a locally free module)**.** An *orientation* of a locally free module $\mathcal{V}$ of constant finite rank r over an F-scheme Y is an isomorphism $o : \det(\mathcal{V}) = \Lambda^r(\mathcal{V}) \to \mathcal{L} \otimes \mathcal{L}$ where $\mathcal{L}$ is an invertible $\mathcal{O}_Y$-module.

Two orientations $o : \det(\mathcal{V}) \to \mathcal{L} \otimes \mathcal{L}, o' : \det(\mathcal{V}) \to \mathcal{L}' \otimes \mathcal{L}'$ are said to be equivalent if there exists an isomorphism $\psi : \mathcal{L} \to \mathcal{L}'$ such that $(\psi \otimes \psi) \circ o = o'$. The equivalence class of o, denoted $\overline{o}$, is called the *orientation class* of o.

Definition 2.2 (Oriented link with two components)**.** An *oriented link* $\mathscr{L}$ with two components is the following data:

- a couple of closed immersions $\varphi_i : \mathbb{A}^2_F \setminus \{0\} \to X$ with disjoint images Z_i;
- for $i \in \{1,2\}$, an orientation class $\overline{o_i}$ of the normal sheaf $\mathcal{N}_{Z_i/X}$, represented by an isomorphism $o_i : \nu_{Z_i} := \det(\mathcal{N}_{Z_i/X}) \to \mathcal{L}_i \otimes \mathcal{L}_i$.

We denote $Z := Z_1 \sqcup Z_2$, $\nu_Z := \det(\mathcal{N}_{Z/X})$.

Remark 2.3. The canonical morphisms $\psi_i : Z_i \to Z$ induce an isomorphism

$$\psi_1^* \oplus \psi_2^* : H^i(Z, \underline{K}_j^{\mathrm{MW}}\{\nu_Z\}) \to H^i(Z_1, \underline{K}_j^{\mathrm{MW}}\{\nu_{Z_1}\}) \oplus H^i(Z_2, \underline{K}_j^{\mathrm{MW}}\{\nu_{Z_2}\})$$

which allows us to identify $H^i(Z, \underline{K}_j^{\mathrm{MW}}\{\nu_Z\})$ with $H^i(Z_1, \underline{K}_j^{\mathrm{MW}}\{\nu_{Z_1}\}) \oplus H^i(Z_2, \underline{K}_j^{\mathrm{MW}}\{\nu_{Z_2}\})$.

Definition 2.4 (Oriented fundamental class and cycles)**.** Let $\mathscr{L}$ be an oriented link with two components and $i \in \{1,2\}$. The *oriented fundamental class* of the i-th component of $\mathscr{L}$, denoted by $[o_i]$, is the unique element of $H^0(Z_i, \underline{K}_{-1}^{\mathrm{MW}}\{\nu_{Z_i}\})$ which is sent to $\eta \in H^0(Z_i, \underline{K}_{-1}^{\mathrm{MW}})$ by the isomorphism $H^0(Z_i, \underline{K}_{-1}^{\mathrm{MW}}\{\nu_{Z_i}\}) \to H^0(Z_i, \underline{K}_{-1}^{\mathrm{MW}})$ induced by o_i (see Lemma B.12).

Furthermore, an *oriented fundamental cycle* of the i-th component of $\mathscr{L}$ is a representative in $\mathcal{C}^0(Z_i, \underline{K}_{-1}^{\mathrm{MW}}\{\nu_{Z_i}\})$ of the oriented fundamental class $[o_i]$.

Remark 2.5. Note that if o_i and o_i' represent the same orientation class then the isomorphism $H^0(Z_i, \underline{K}_{-1}^{\mathrm{MW}}\{\nu_{Z_i}\}) \to H^0(Z_i, \underline{K}_{-1}^{\mathrm{MW}})$ induced by o_i' is the same as the one induced by o_i, hence the oriented fundamental class $[o_i]$ only depends on the orientation class $\overline{o_i}$.

Recall that the boundary map $\partial : H^1(X \setminus Z, \underline{K}_1^{\mathrm{MW}}) \to H^0(Z, \underline{K}_{-1}^{\mathrm{MW}}\{\nu_Z\})$ (see Definition B.10) is an isomorphism (see the localization long exact sequence (in Theorem B.11) and note that the groups $H^1(X, \underline{K}_1^{\mathrm{MW}})$ and $H^2(X, \underline{K}_1^{\mathrm{MW}})$ vanish (see Proposition B.13)).

Definition 2.6 (Seifert class and Seifert divisors)**.** Let $\mathscr{L}$ be an oriented link with two components. The couple of *Seifert classes* of $\mathscr{L}$ is the couple $(\mathcal{S}_{o_1}, \mathcal{S}_{o_2})$, or $(\mathcal{S}_1, \mathcal{S}_2)$ for short, of elements of $H^1(X \setminus Z, \underline{K}_1^{\mathrm{MW}})$ such that $\partial(\mathcal{S}_1) = ([o_1], 0)$ and $\partial(\mathcal{S}_2) = (0, [o_2])$.

For $i \in \{1,2\}$, we call $\mathcal{S}_i$ the *Seifert class* of Z_i relative to the link $\mathscr{L}$. Furthermore, a *Seifert divisor* of Z_i relative to the link $\mathscr{L}$ is a representative in $\mathcal{C}^1(X \setminus Z, \underline{K}_1^{\mathrm{MW}})$ of $\mathcal{S}_i$.

Remark 2.7. For $i \in \{1,2\}$, the Seifert class of Z_i relative to $\mathscr{L}$ depends on Z and not only on Z_i (and its orientation class $\overline{o_i}$). We could define a weaker notion of Seifert class of Z_i, which would only depend on Z_i (and $\overline{o_i}$), but it is important for what follows to have this stronger notion of Seifert class.

See Appendix B.3 for recollections on the intersection product.

Definition 2.8 (Quadratic linking class)**.** Let $\mathscr{L}$ be an oriented link with two components. The *quadratic linking class* of $\mathscr{L}$, denoted by $\mathrm{Qlc}_{\mathscr{L}}$, is the image of the intersection product of the Seifert class $\mathcal{S}_1$ with the Seifert class $\mathcal{S}_2$ by the boundary map $\partial : H^2(X \setminus Z, \underline{K}_2^{\mathrm{MW}}) \to H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\nu_Z\})$. We denote $\mathrm{Qlc}_{\mathscr{L}} = (\sigma_{1,\mathscr{L}}, \sigma_{2,\mathscr{L}}) \in H^1(Z_1, \underline{K}_0^{\mathrm{MW}}\{\nu_{Z_1}\}) \oplus H^1(Z_2, \underline{K}_0^{\mathrm{MW}}\{\nu_{Z_2}\})$ (see Remark 2.3).

Remark 2.9. Note that the quadratic linking class $\mathrm{Qlc}_{\mathscr{L}}$ contains as much information as the intersection product $\mathcal{S}_1 \cdot \mathcal{S}_2$ since the boundary map $\partial : H^2(X \setminus Z, \underline{K}_2^{\mathrm{MW}}) \to H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\nu_Z\})$ is injective (see the localization long exact sequence (in Theorem B.11) and note that the group $H^2(X, \underline{K}_2^{\mathrm{MW}})$ vanishes (see Proposition B.13)). Also note that $\mathrm{Qlc}_{\mathscr{L}} \in \ker(i_*)$ since the image of ∂ is the kernel of $i_* : H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\nu_Z\}) \to H^3(X, \underline{K}_2^{\mathrm{MW}})$, where i_* is the push-forward of the closed immersion $i : Z \to X$ (see the localization long exact sequence (in Theorem B.11)).

Notation 2.10. For $i \in \{1,2\}$, we denote by $\widetilde{o_i}$ the isomorphism $H^1(Z_i, \underline{K}_0^{\mathrm{MW}}\{\nu_{Z_i}\}) \to H^1(Z_i, \underline{K}_0^{\mathrm{MW}})$ induced by o_i (see Lemma B.12) and by φ_i^* the isomorphism $H^1(Z_i, \underline{K}_0^{\mathrm{MW}}) \to H^1(\mathbb{A}_F^2 \setminus \{0\}, \underline{K}_0^{\mathrm{MW}})$ induced by φ_i.

Recall that we fixed an isomorphism $\zeta : H^1(\mathbb{A}_F^2 \setminus \{0\}, \underline{K}_0^{\mathrm{MW}}) \to \mathrm{W}(F)$ (see Subsection 2.1).

Definition 2.11 (Quadratic linking degree)**.** Let $\mathscr{L}$ be an oriented link with two components. The *quadratic linking degree* of $\mathscr{L}$, denoted by $\mathrm{Qld}_{\mathscr{L}}$, is the image of the quadratic linking class of $\mathscr{L}$ by the isomorphism $(\zeta \oplus \zeta) \circ (\varphi_1^* \oplus \varphi_2^*) \circ (\widetilde{o_1} \oplus \widetilde{o_2}) : H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\nu_Z\}) \to \mathrm{W}(F) \oplus \mathrm{W}(F)$.

2.3. **Invariants of the quadratic linking degree.** By construction, the quadratic linking degree depends on choices of orientations and of parametrizations of $\mathbb{A}_F^2 \setminus \{0\} \to X$. In this Subsection we determine how it depends on such choices and construct invariants from the quadratic linking degree.

Throughout this Subsection, $\mathscr{L}$ is an oriented link with two components and we denote $\mathrm{Qld}_{\mathscr{L}} = (d_1, d_2) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$.

We start by recalling how orientation classes can change.

Lemma 2.12. *Let $i \in \{1,2\}$ and $o'_i : \nu_{Z_i} \to \mathcal{L}'_i \otimes \mathcal{L}'_i$ be an orientation of the normal sheaf of Z_i in X. There exists $a \in F^*$ such that the orientation class of o'_i is the orientation class of $o_i \circ (\times a)$.*

Proof. Recall that every invertible $\mathcal{O}_{\mathbb{A}^2_F}$-module is isomorphic to $\mathcal{O}_{\mathbb{A}^2_F}$ (since $\mathbb{A}^2_F$ is factorial) and that every invertible $\mathcal{O}_{\mathbb{A}^2_F\setminus\{0\}}$-module is the restriction of an invertible $\mathcal{O}_{\mathbb{A}^2_F}$-module hence every invertible $\mathcal{O}_{\mathbb{A}^2_F\setminus\{0\}}$-module is isomorphic to $\mathcal{O}_{\mathbb{A}^2_F\setminus\{0\}}$. Since $Z_i \simeq \mathbb{A}^2_F \setminus \{0\}$, there exist isomorphisms $\psi : \mathcal{L}_i \to \mathcal{O}_{Z_i}$ and $\psi' : \mathcal{L}'_i \to \mathcal{O}_{Z_i}$. From Definition 2.1, $\overline{(\psi \otimes \psi) \circ o_i} = \overline{o_i}$ and $\overline{(\psi' \otimes \psi') \circ o'_i} = \overline{o'_i}$. Denoting by $m : \mathcal{O}_{Z_i} \otimes \mathcal{O}_{Z_i} \to \mathcal{O}_{Z_i}$ the multiplication, the morphism $m \circ ((\psi' \otimes \psi') \circ o'_i) \circ ((\psi \otimes \psi) \circ o_i)^{-1} \circ m^{-1}$ is an automorphism of $\mathcal{O}_{Z_i}$ hence is the multiplication by an element of $\Gamma(Z_i, \mathcal{O}^*_{Z_i})$, i.e. by an element of F^*. The result follows directly. □

Recall that automorphisms of $\mathbb{A}^2_F \setminus \{0\}$ are restrictions of automorphisms of $\mathbb{A}^2_F$ which preserve the origin, hence they induce changes of coordinates of $\mathbb{A}^2_F = \mathrm{Spec}(F[u,v])$. We denote by J_ψ the Jacobian determinant of an automorphism ψ of $\mathbb{A}^2_F \setminus \{0\}$; note that J_ψ is in F^* since $(F[u,v])^* = F^*$.

Lemma 2.13. *Let $a = (a_1, a_2)$ be a couple of elements of F^* and $\psi = (\psi_1, \psi_2)$ be a couple of automorphisms of $\mathbb{A}^2_F \setminus \{0\}$.*

1. *Let $\mathscr{L}_a$ be the link obtained from $\mathscr{L}$ by changing the orientation o_1 into $o_1 \circ (\times a_1)$ and the orientation o_2 into $o_2 \circ (\times a_2)$. Then $\mathrm{Qlc}_{\mathscr{L}_a} = \langle a_1 a_2 \rangle \,\mathrm{Qlc}_{\mathscr{L}}$ and $\mathrm{Qld}_{\mathscr{L}_a} = (\langle a_2 \rangle d_1, \langle a_1 \rangle d_2)$.*
2. *Let $\mathscr{L}_\psi$ be the link obtained from $\mathscr{L}$ by changing $\varphi_1 : \mathbb{A}^2_F \setminus \{0\} \to X$ into $\varphi_1 \circ \psi_1$ and $\varphi_2 : \mathbb{A}^2_F \setminus \{0\} \to X$ into $\varphi_2 \circ \psi_2$. Then $\mathrm{Qlc}_{\mathscr{L}_\psi} = \mathrm{Qlc}_{\mathscr{L}}$ and $\mathrm{Qld}_{\mathscr{L}_\psi} = (\langle J_{\psi_1} \rangle d_1, \langle J_{\psi_2} \rangle d_2)$.*
3. *Let $\mathscr{L}'$ be the link obtained from $\mathscr{L}$ by changing the order of the components. Then $\mathrm{Qlc}_{\mathscr{L}'} = -\mathrm{Qlc}_{\mathscr{L}}$ and $\mathrm{Qld}_{\mathscr{L}'} = (-d_2, -d_1)$.*

Proof. (1) Note that for all $i \in \{1,2\}$, $[o_i \circ (\times a_i)] = \langle a_i^{-1} \rangle [o_i] = \langle a_i \rangle [o_i]$ hence, by Proposition A.3 and Proposition B.16:

$$\begin{aligned} \mathcal{S}_{o_1\circ(\times a_1)} = \langle a_1 \rangle \mathcal{S}_{o_1} &\text{ and } \mathcal{S}_{o_2\circ(\times a_2)} = \langle a_2 \rangle \mathcal{S}_{o_2} \\ \mathcal{S}_{o_1\circ(\times a_1)} \cdot \mathcal{S}_{o_2\circ(\times a_2)} &= \langle a_1 a_2 \rangle \mathcal{S}_{o_1} \cdot \mathcal{S}_{o_2} \\ \partial(\mathcal{S}_{o_1\circ(\times a_1)} \cdot \mathcal{S}_{o_2\circ(\times a_2)}) &= \langle a_1 a_2 \rangle \partial(\mathcal{S}_{o_1} \cdot \mathcal{S}_{o_2}) \\ \mathrm{Qlc}_{\mathscr{L}_a} &= \langle a_1 a_2 \rangle \,\mathrm{Qlc}_{\mathscr{L}} \end{aligned}$$

Note that $\widetilde{o_1 \circ (\times a_1)}(\langle a_1 a_2 \rangle \sigma_{1,\mathscr{L}}) = \langle a_1 \rangle \widetilde{o_1}(\langle a_1 a_2 \rangle \sigma_{1,\mathscr{L}}) = \langle a_1^2 a_2 \rangle \widetilde{o_1}(\sigma_{1,\mathscr{L}}) = \langle a_2 \rangle \widetilde{o_1}(\sigma_{1,\mathscr{L}})$ and similarly $\widetilde{o_2 \circ (\times a_2)}(\langle a_1 a_2 \rangle \sigma_{2,\mathscr{L}}) = \langle a_1 \rangle \widetilde{o_2}(\sigma_{2,\mathscr{L}})$. It follows that $\mathrm{Qld}_{\mathscr{L}_a} = (\langle a_2 \rangle d_1, \langle a_1 \rangle d_2)$.

(2) From the definitions, $\mathrm{Qlc}_{\mathscr{L}_\psi} = \mathrm{Qlc}_{\mathscr{L}}$ and $(\widetilde{o_1} \oplus \widetilde{o_2})(\mathrm{Qlc}_{\mathscr{L}_\psi}) = (\widetilde{o_1} \oplus \widetilde{o_2})(\mathrm{Qlc}_{\mathscr{L}})$. Let $i \in \{1,2\}$. We denote by $\psi_i^* : H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}_0^{\mathrm{MW}}) \to H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}_0^{\mathrm{MW}})$ the isomorphism induced by ψ_i. Note that $(\varphi_i \circ \psi_i)^*(\widetilde{o_i}(\sigma_{i,\mathscr{L}})) = \psi_i^*(\varphi_i^*(\widetilde{o_i}(\sigma_{i,\mathscr{L}})))$ and that the following diagram is commutative:

$$\begin{array}{ccc} H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}_0^{\mathrm{MW}}) & \xrightarrow{\partial} & H^0(\{0\}, \underline{K}_{-2}^{\mathrm{MW}}\{\det(\mathcal{N}_{\overline{\{0\}}/\mathbb{A}^2_F})\}) \\ \downarrow \psi_i^* & & \downarrow \psi_i^* \\ H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}_0^{\mathrm{MW}}) & \xrightarrow[\partial]{} & H^0(\{0\}, \underline{K}_{-2}^{\mathrm{MW}}\{\det(\mathcal{N}_{\overline{\{0\}}/\mathbb{A}^2_F})\}) \end{array}$$

Hence $\partial((\varphi_i \circ \psi_i)^*(\widetilde{o_i}(\sigma_{i,\mathscr{L}}))) = \psi_i^*(\partial(\varphi_i^*(\widetilde{o_i}(\sigma_{i,\mathscr{L}}))))$.

Finally note that for all $\alpha \in K_{-2}^{\mathrm{MW}}(F)$, $\psi_i^*(\alpha \otimes (\overline{u}^* \wedge \overline{v}^*)) = \langle J_{\psi_i} \rangle \alpha \otimes (\overline{u}^* \wedge \overline{v}^*)$. It follows from Definition B.15 that $\mathrm{Qld}_{\mathscr{L}_\psi} = (\langle J_{\psi_1} \rangle d_1, \langle J_{\psi_2} \rangle d_2)$.

(3) By Proposition B.17, $\mathcal{S}_2 \cdot \mathcal{S}_1 = \langle -1 \rangle (\mathcal{S}_1 \cdot \mathcal{S}_2)$ hence by Proposition A.3, $\partial(\mathcal{S}_2 \cdot \mathcal{S}_1) = \langle -1 \rangle \partial(\mathcal{S}_1 \cdot \mathcal{S}_2) = -\partial(\mathcal{S}_1 \cdot \mathcal{S}_2)$ hence $\mathrm{Qlc}_{\mathscr{L}'} = -\mathrm{Qlc}_{\mathscr{L}}$. It follows that $\mathrm{Qld}_{\mathscr{L}'} = (-d_2, -d_1)$. □

We directly get the following invariants.

Corollary 2.14. *The rank modulo 2 of d_1 and the rank modulo 2 of d_2 are invariant under changes of orientations o_1, o_2 and under changes of $\varphi_1, \varphi_2 : \mathbb{A}^2_F \setminus \{0\} \to X$.*

Proof. For all $a \in F^*$, the rank modulo 2 of an element of the Witt ring $\mathrm{W}(F)$ is invariant under the multiplication by $\langle a \rangle$. The result follows directly from Lemma 2.13. □

Recall that $\mathrm{W}(\mathbb{R}) \simeq \mathbb{Z}$ (via the signature). See Examples 4.3 for an illustration of the use of the following invariant.

Corollary 2.15. *If $F = \mathbb{R}$ then the absolute value of d_1 and the absolute value of d_2 are invariant under changes of orientations o_1, o_2 and under changes of $\varphi_1, \varphi_2 : \mathbb{A}^2_{\mathbb{R}} \setminus \{0\} \to X$.*

Proof. For all $a \in \mathbb{R}^*$, $\langle a \rangle = \langle 1 \rangle = 1$ or $\langle a \rangle = \langle -1 \rangle = -1$ since every real number is a square or the opposite of a square. The result follows directly from Lemma 2.13. □

The following Lemma-Definition is an inductive definition. For each $d \in \mathrm{W}(F)$, with k ranging over the nonnegative even integers, we define an abelian group $Q_{d,k}$ and an element $\Sigma_k(d) \in Q_{d,k}$. In Theorem 2.17 we will see that $\Sigma_k(d_1)$ and $\Sigma_k(d_2)$ are invariants.

Lemma-Definition 2.16. *Let $d = \sum_{i=1}^{n} \langle a_i \rangle \in \mathrm{W}(F)$. There exists a unique sequence of abelian groups $Q_{d,k}$ and of elements $\Sigma_k(d) \in Q_{d,k}$, where k ranges over the nonnegative even integers, such that:*

- *$Q_{d,0} = \mathrm{W}(F)$ and $\Sigma_0(d) = 1 \in Q_{d,0}$;*
- *for each positive even integer k, $Q_{d,k}$ is the quotient group $Q_{d,k-2}/(\Sigma_{k-2}(d))$;*
- *for each positive even integer k, $\Sigma_k(d) = \sum_{1 \le i_1 < \cdots < i_k \le n} \langle \prod_{1 \le j \le k} a_{i_j} \rangle \in Q_{d,k}$.*

Proof. Recall the following presentation of the abelian group $\mathrm{W}(F)$: its generators are the $\langle a \rangle$ for $a \in F^*$ and its relations are the following:

(1) $\langle ab^2 \rangle = \langle a \rangle$ for all $a, b \in F^*$;
(2) $\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle (a+b)ab \rangle$ for all $a, b \in F^*$ such that $a + b \neq 0$;
(3) $\langle 1 \rangle + \langle -1 \rangle = 0$.

We denote by G the free abelian group of generators the $\langle a \rangle$ for $a \in F^*$, by G_1 the quotient of G by the first relation above and by G_2 the quotient of G_1 by the second relation above.

Let k be a nonnegative even integer such that for all nonnegative even integers $l < k$, $Q_{d,l}$ is an abelian group and $\Sigma_l(d) \in Q_{d,l}$ which verify the conditions of the statement. Note that the quotient of the abelian group $Q_{d,k-2}$ by its subgroup $(\Sigma_{k-2}(d))$ is well-defined, so we can fix $Q_{d,k} = Q_{d,k-2}/(\Sigma_{k-2}(d))$. Let $\sum_{i=1}^{n} \langle a_i \rangle \in G$. Note that $\sum_{1 \le i_1 < \cdots < i_k \le n} \langle \prod_{1 \le j \le k} a_{i_j} \rangle \in Q_{d,k}$ is well-defined (since it is well-defined in G and $Q_{d,k}$ is obtained from G by quotienting several times). In fact, $\sum_{1 \le i_1 < \cdots < i_k \le n} \langle \prod_{1 \le j \le k} a_{i_j} \rangle \in Q_{d,k}$ depends only on the class of $\sum_{i=1}^{n} \langle a_i \rangle$ in G_1 since for all $b \in F^*$, $\sum_{2 \le i_2 < \cdots < i_k \le n} \langle a_1 b^2 \prod_{2 \le j \le k} a_{i_j} \rangle + \sum_{2 \le i_1 < \cdots < i_k \le n} \langle \prod_{1 \le j \le k} a_{i_j} \rangle = \sum_{1 \le i_1 < \cdots < i_k \le n} \langle \prod_{1 \le j \le k} a_{i_j} \rangle \in Q_{d,k-2}$ (since this equality is already true in $\mathrm{W}(F)$ and $Q_{d,k}$ is obtained from $\mathrm{W}(F)$ by quotienting several times) and similarly for other indices.

Furthermore, $\sum_{1\le i_1<\cdots<i_k\le n}\langle\prod_{1\le j\le k}a_{i_j}\rangle\in Q_{d,k}$ depends only on the class of $\sum_{i=1}^{n}\langle a_i\rangle$ in G_2 since if $a_1+a_2\neq 0$ then in $Q_{d,k}$:

$$\begin{aligned}
&\sum_{3\le i_3<\cdots<i_k\le n}\langle (a_1+a_2)^2a_1a_2\prod_{3\le j\le k}a_{i_j}\rangle+\sum_{3\le i_1<\cdots<i_k\le n}\langle\prod_{1\le j\le k}a_{i_j}\rangle\\
&+\sum_{3\le i_2<\cdots<i_k\le n}\langle (a_1+a_2)\prod_{2\le j\le k}a_{i_j}\rangle+\sum_{3\le i_2<\cdots<i_k\le n}\langle (a_1+a_2)a_1a_2\prod_{2\le j\le k}a_{i_j}\rangle\\
&=\sum_{3\le i_3<\cdots<i_k\le n}\langle a_1a_2\prod_{3\le j\le k}a_{i_j}\rangle+\sum_{3\le i_1<\cdots<i_k\le n}\langle\prod_{1\le j\le k}a_{i_j}\rangle\\
&+(\langle a_1+a_2\rangle+\langle (a_1+a_2)a_1a_2\rangle)\sum_{3\le i_2<\cdots<i_k\le n}\langle (a_1+a_2)\prod_{2\le j\le k}a_{i_j}\rangle\\
&=\sum_{3\le i_3<\cdots<i_k\le n}\langle a_1a_2\prod_{3\le j\le k}a_{i_j}\rangle+\sum_{3\le i_1<\cdots<i_k\le n}\langle\prod_{1\le j\le k}a_{i_j}\rangle\\
&+(\langle a_1\rangle+\langle a_2\rangle)\sum_{3\le i_2<\cdots<i_k\le n}\langle (a_1+a_2)\prod_{2\le j\le k}a_{i_j}\rangle\\
&=\sum_{1\le i_1<\cdots<i_k\le n}\langle\prod_{1\le j\le k}a_{i_j}\rangle
\end{aligned}$$

(since these equalities are already true in $\mathrm{W}(F)$) and similarly for other indices.

Finally, $\sum_{1\le i_1<\cdots<i_k\le n}\langle\prod_{1\le j\le k}a_{i_j}\rangle\in Q_{d,k}$ depends only on the class of $\sum_{i=1}^{n}\langle a_i\rangle$ in $\mathrm{W}(F)$ (i.e. on d) since, with the convention that $\sum_{1\le i_3<\cdots<i_2\le n}\langle\prod_{3\le j\le 2}a_{i_j}\rangle=1$,

$$\sum_{1\le i_1<\cdots<i_k\le n}\langle\prod_{1\le j\le k}a_{i_j}\rangle+(\langle 1\rangle+\langle -1\rangle)\sum_{1\le i_2<\cdots<i_k\le n}\langle\prod_{2\le j\le k}a_{i_j}\rangle+\langle -1\rangle\sum_{1\le i_3<\cdots<i_k\le n}\langle\prod_{3\le j\le k}a_{i_j}\rangle$$

is equal to $\sum_{1\le i_1<\cdots<i_k\le n}\langle\prod_{1\le j\le k}a_{i_j}\rangle-\Sigma_{k-2}(d)$ in $Q_{d,k}$ (since this equality is already true in $\mathrm{W}(F)$) which is equal to $\sum_{1\le i_1<\cdots<i_k\le n}\langle\prod_{1\le j\le k}a_{i_j}\rangle$ in $Q_{d,k}=Q_{d,k-2}/(\Sigma_{k-2}(d))$. Thus we can fix $\Sigma_k(d)=\sum_{1\le i_1<\cdots<i_k\le n}\langle\prod_{1\le j\le k}a_{i_j}\rangle\in Q_{d,k}$. □

It follows from Lemma-Definition 2.16 that we have a map $\Sigma_k:\mathrm{W}(F)\to\bigcup_{d\in\mathrm{W}(F)}Q_{d,k}$ which verifies that for all $d\in\mathrm{W}(F)$, $\Sigma_k(d)\in Q_{d,k}$. This provides new invariants of the quadratic linking degree. See Examples 4.2 for an illustration of the use of $\Sigma_2:\mathrm{W}(F)\to\mathrm{W}(F)/(1)$.

Theorem 2.17. *Let $\mathscr{L}$ be an oriented link with two components and k be a positive even integer. We denote the quadratic linking degree of $\mathscr{L}$ by $\mathrm{Qld}_{\mathscr{L}}=(d_1,d_2)\in\mathrm{W}(F)\oplus\mathrm{W}(F)$. Then $\Sigma_k(d_1)$ and $\Sigma_k(d_2)$ are invariant under changes of orientations o_1,o_2 and under changes of $\varphi_1,\varphi_2:\mathbb{A}^2_F\setminus\{0\}\to X$.*

Proof. Let $\sum_{1\le i\le n}\langle a_i\rangle\in\mathrm{W}(F)$. Note that for all $b\in F^*$:

$$\Sigma_k(\langle b\rangle\sum_{1\le i\le n}\langle a_i\rangle)=\sum_{1\le i_1<\cdots<i_k\le n}\langle b^k\prod_{1\le j\le k}a_{i_j}\rangle=\sum_{1\le i_1<\cdots<i_k\le n}\langle\prod_{1\le j\le k}a_{i_j}\rangle=\Sigma_k(\sum_{1\le i\le n}\langle a_i\rangle)$$

since b^k is a square as k is even. The result follows directly from Lemma 2.13. □

3. How to compute the quadratic linking degree

In this Section, we give a method to compute the quadratic linking class and the quadratic linking degree under reasonable assumptions on the link (which are verified in the examples of Section 4). See Subsection 2.1 for notations and Subsection 2.2 for definitions.

3.1. Assumptions.

Let $\mathscr{L}$ be an oriented link with two components such that for all $i \in \{1,2\}$, the closure $\overline{Z_i} \subset \mathbb{A}^4_F$ of Z_i is given by two equations

$$f_i(x,y,z,t) = 0, g_i(x,y,z,t) = 0$$

with f_i and g_i irreducible. We also assume that the subscheme of X given by the equations $g_1 = 0$ and $g_2 = 0$ is of codimension 2 in X.

Let $i \in \{1,2\}$. Note that we can define an orientation of $\mathcal{N}_{Z_i/X}$ from the (ordered) couple (f_i, g_i). Indeed, $\mathcal{N}_{Z_i/X}$ is the dual of the conormal sheaf $\mathcal{C}_{Z_i/X} = \mathscr{I}_{Z_i}/\mathscr{I}^2_{Z_i}$, where $\mathscr{I}_{Z_i}$ is the ideal sheaf of Z_i in X, and we have the following short exact sequence:

$$0 \longrightarrow (\mathcal{C}_{V(g_i)/\mathbb{A}^4_F})_{|Z_i} \longrightarrow \mathcal{C}_{Z_i/X} = (\mathcal{C}_{\overline{Z_i}/\mathbb{A}^4_F})_{|Z_i} \longrightarrow (\mathcal{C}_{V(f_i)/\mathbb{A}^4_F})_{|Z_i} \longrightarrow 0$$

We define the orientation $o_{(f_i,g_i)}$ as the isomorphism $\nu_{Z_i} \to \mathcal{O}_{Z_i} \otimes \mathcal{O}_{Z_i}$ which sends $\overline{f_i}^* \wedge \overline{g_i}^*$ to $1 \otimes 1$. By Lemma 2.12, there exists $a_i \in F^*$ such that $\overline{o_i} = \overline{o_{(f_i,g_i)} \circ (\times a_i)} = \overline{o_{(a_i^{-1} f_i, g_i)}}$. Without loss of generality (since we can replace f_i with $a_i^{-1} f_i$), we assume that $o_i = o_{(f_i,g_i)}$.

3.2. Notations.

We denote by $\chi^{\mathrm{odd}} : \mathbb{Z} \to \{0,1\}$ the characteristic function of the set of odd numbers.

We denote $\epsilon := -\langle -1 \rangle$ and for all $n \in \mathbb{N}_0$, $n_\epsilon := \displaystyle\sum_{i=1}^{n} \langle (-1)^{i-1} \rangle$ and $(-n)_\epsilon := \epsilon\, n_\epsilon$.

In order to make explicit computations, we introduce the following notations. Note that the quadratic linking class and degree of $\mathscr{L}$ which are computed in Theorems 3.1 and 3.2 respectively do not depend on the choices of uniformizing parameters made below (see Definitions 2.8 and 2.11).

We denote by I the set of generic points of irreducible components of the subscheme of $X \setminus Z$ given by the equations $g_1 = 0$ and $g_2 = 0$.

For every $p \in I$, we denote by π_p a uniformizing parameter of the discrete valuation ring $\mathcal{O}_{X\setminus Z,p}/(g_1)$, by u_p a unit in $\mathcal{O}_{X\setminus Z,p}/(g_1)$ and by $m_p \in \mathbb{Z}$ such that $g_2 = u_p \pi_p^{m_p} \in \mathcal{O}_{X\setminus Z,p}/(g_1)$.

For every $p \in I$ and $q \in \overline{\{p\}}^{(1)} \cap Z$, we denote by $\pi_{p,q}$ a uniformizing parameter of the discrete valuation ring $\mathcal{O}_{\overline{\{p\}},q}$, by $u_{p,q}$ a unit in $\mathcal{O}_{\overline{\{p\}},q}$ and by $m_{p,q} \in \mathbb{Z}$ such that $f_1 f_2 u_p = u_{p,q} \pi_{p,q}^{m_{p,q}} \in \mathcal{O}_{\overline{\{p\}},q}$.

For every $i \in \{1,2\}$, $p \in I$ and $q \in \overline{\{p\}}^{(1)} \cap Z_i$, we denote by $\tau_{p,q} \in \nu_q$ such that $\overline{\pi_{p,q}}^* \otimes \overline{\pi_p}^* \otimes \overline{g_1}^* = \tau_{p,q} \otimes (\overline{f_i}^* \wedge \overline{g_i}^*)$, by $v_{p,q,0}$ the discrete valuation of $\mathcal{O}_{\overline{\{\varphi_i^{-1}(q)\}},0}$ and by $\pi_{p,q,0}$ a uniformizing parameter for $v_{p,q,0}$. Note that such a $\tau_{p,q}$ exists since $\overline{\pi_{p,q}}^* \otimes \overline{\pi_p}^* \otimes \overline{g_1}^* \in \mathbb{Z}[(\nu_{p,q} \otimes_{\kappa(q)} (\nu_{Z_i})_{|q}) \setminus \{0\}]$.

For every $i \in \{1,2\}$, $p \in I$ and $q \in \overline{\{p\}}^{(1)} \cap Z_i$, we let $(u_{p,q,0}, m_{p,q,0}) \in \mathcal{O}^*_{\overline{\{\varphi_i^{-1}(q)\}},0} \times \mathbb{Z}$ be the unique couple such that $\varphi_i^*(\overline{u_{p,q}}) = u_{p,q,0} \pi_{p,q,0}^{m_{p,q,0}}$ and we denote by $\lambda_{p,q,0} \in K_0^{\mathrm{MW}}(F)$ such that $\eta^2 \otimes (\overline{\pi_{p,q,0}}^* \otimes \varphi_i^*(\tau_{p,q})) = \lambda_{p,q,0}\, \eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$. Note that such a $\lambda_{p,q,0}$ exists since $\overline{\pi_{p,q,0}}^* \otimes \varphi_i^*(\tau_{p,q}) \in \mathbb{Z}[(\det(\mathcal{N}_{\{0\}/\mathbb{A}^2_F})_{|0}) \setminus \{0\}]$.

3.3. Computing the quadratic linking class and degree.

Theorem 3.1. *Under the assumptions of Subsection 3.1 and with the notations in Subsection 3.2, the cycle*

$$\sum_{p\in I} \sum_{q \in \overline{\{p\}}^{(1)} \cap Z} \langle \overline{u_{p,q}} \rangle \eta\, \chi^{\mathrm{odd}}(m_p m_{p,q}) \otimes (\overline{\pi_{p,q}}^* \otimes \overline{\pi_p}^* \otimes \overline{g_1}^*)$$

where $\langle \overline{u_{p,q}} \rangle \eta\, \chi^{\mathrm{odd}}(m_p m_{p,q}) \otimes (\overline{\pi_{p,q}}^* \otimes \overline{\pi_p}^* \otimes \overline{g_1}^*) \in K_{-1}^{\mathrm{MW}}(\kappa(q), \nu_q \otimes (\nu_Z)_{|q})$, *represents the quadratic linking class of* $\mathscr{L}$.

Proof. From Definition 2.4, the oriented fundamental class $[o_i]$ is the class in $H^0(Z_i, \underline{K}_{-1}^{\mathrm{MW}}\{\nu_{Z_i}\})$ of $\eta \otimes (\overline{f_i}^* \wedge \overline{g_i}^*)$. It follows from Definition 2.6 and Theorem A.11 that the Seifert class $\mathcal{S}_i$ of Z_i is the class in $H^1(X \setminus Z, \underline{K}_1^{\mathrm{MW}})$ of $\langle f_i \rangle \otimes \overline{g_i}^*$ (over the generic point p_i of the hypersurface of $X \setminus Z$ of equation $g_i = 0$). In the expression above, $\langle f_i \rangle \in K_0^{\mathrm{MW}}(\kappa(p_i))$ and $\overline{g_i}^* \in \mathbb{Z}[\det(\mathcal{N}_{\overline{\{p_i\}}/X \setminus Z}) \setminus \{0\}]$; with a slight abuse of notation, we denoted by f_i the image in the fraction field of $F[x, y, z, t]/(g_i)$ of $f_i \in F[x, y, z, t]$. We will make similar slight abuses of notation below.

By Corollary B.19, the intersection product of the Seifert class $\mathcal{S}_1$ of Z_1 with the Seifert class $\mathcal{S}_2$ of Z_2 is the class in $H^2(X \setminus Z, \underline{K}_2^{\mathrm{MW}})$ of the cycle:

$$\sum_{p \in I} (m_p)_\epsilon \langle f_1 f_2 u_p \rangle \otimes (\overline{\pi_p}^* \otimes \overline{g_1}^*)$$

The quadratic linking class is the image of this intersection product by the boundary map $\partial : H^2(X \setminus Z, \underline{K}_2^{\mathrm{MW}}) \to H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\nu_Z\})$ thus the cycle

$$\sum_{p \in I} \sum_{q \in \overline{\{p\}}^{(1)} \cap Z} (m_p)_\epsilon \partial_{v_q}^{\pi_{p,q}}(\langle f_1 f_2 u_p \rangle) \otimes (\overline{\pi_{p,q}}^* \otimes \overline{\pi_p}^* \otimes \overline{g_1}^*)$$

represents the quadratic linking class (note that we used Proposition A.3 to extract $(m_p)_\epsilon$ from the morphism $\partial_{v_q}^{\pi_{p,q}}$). By Theorem A.11 and Lemma A.10, the cycle

$$\sum_{p \in I} \sum_{q \in \overline{\{p\}}^{(1)} \cap Z} \langle \overline{u_{p,q}} \rangle \eta\, \chi^{\mathrm{odd}}(m_p m_{p,q}) \otimes (\overline{\pi_{p,q}}^* \otimes \overline{\pi_p}^* \otimes \overline{g_1}^*)$$

represents the quadratic linking class of $\mathscr{L}$. □

Theorem 3.2. *Under the assumptions of Subsection 3.1 and with the notations in Subsection 3.2, the quadratic linking degree of* $\mathscr{L}$ *is the following couple of elements of* $\mathrm{W}(F)$*:*

$$\left(\sum_{p \in I} \sum_{q \in \overline{\{p\}}^{(1)} \cap Z_1} \lambda_{p,q,0} \langle \overline{u_{p,q,0}} \rangle\, \chi^{\mathrm{odd}}(m_p m_{p,q} m_{p,q,0}), \sum_{p \in I} \sum_{q \in \overline{\{p\}}^{(1)} \cap Z_2} \lambda_{p,q,0} \langle \overline{u_{p,q,0}} \rangle\, \chi^{\mathrm{odd}}(m_p m_{p,q} m_{p,q,0}) \right)$$

Proof. Recall from Definition 2.11 that the first step in computing the quadratic linking degree from the quadratic linking class consists in applying $\widetilde{o_1} \oplus \widetilde{o_2}$. It follows from Theorem 3.1 and the assumption that for all $i \in \{1, 2\}$, $o_i = o_{(f_i, g_i)}$ (see Subsection 3.1) that the couple of cycles

$$\left(\sum_{p \in I} \sum_{q \in \overline{\{p\}}^{(1)} \cap Z_1} \langle \overline{u_{p,q}} \rangle \eta\, \chi^{\mathrm{odd}}(m_p m_{p,q}) \otimes \tau_{p,q}, \sum_{p \in I} \sum_{q \in \overline{\{p\}}^{(1)} \cap Z_2} \langle \overline{u_{p,q}} \rangle \eta\, \chi^{\mathrm{odd}}(m_p m_{p,q}) \otimes \tau_{p,q} \right)$$

where $\langle \overline{u_{p,q}} \rangle \eta\, \chi^{\mathrm{odd}}(m_p m_{p,q}) \otimes \tau_{p,q} \in K_{-1}^{\mathrm{MW}}(\kappa(q), \nu_{p,q})$, represents $(\widetilde{o_1} \oplus \widetilde{o_2})(\mathrm{Qlc}_{\mathscr{L}})$.

It follows that the couple of cycles

$$\left(\sum_{p \in I} \sum_{q \in \overline{\{p\}}^{(1)} \cap Z_1} \langle \varphi_1^*(\overline{u_{p,q}}) \rangle \eta\, \chi^{\mathrm{odd}}(m_p m_{p,q}) \otimes \varphi_1^*(\tau_{p,q}), \right.$$

$$\left. \sum_{p \in I} \sum_{q \in \overline{\{p\}}^{(1)} \cap Z_2} \langle \varphi_2^*(\overline{u_{p,q}}) \rangle \eta\, \chi^{\mathrm{odd}}(m_p m_{p,q}) \otimes \varphi_2^*(\tau_{p,q}) \right)$$

where for all $i \in \{1, 2\}$, $\langle \varphi_i^*(\overline{u_{p,q}}) \rangle \eta\, \chi^{\mathrm{odd}}(m_p m_{p,q}) \otimes \varphi_i^*(\tau_{p,q}) \in K_{-1}^{\mathrm{MW}}(\kappa(\varphi_i^{-1}(q)), \nu_{\varphi_i^{-1}(q)})$, represents $(\varphi_1^* \oplus \varphi_2^*)(\widetilde{o_1} \oplus \widetilde{o_2})(\mathrm{Qlc}_{\mathscr{L}})$. This is the second step in computing the quadratic linking degree (see Definition 2.11).

Recall from Definition 2.11 and Definition B.15 that the third step in computing the quadratic linking degree consists in applying the boundary map

$$\partial : \mathcal{C}^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_0) \to \mathcal{C}^0(\{0\}, \underline{K}^{\mathrm{MW}}_{-2}\{\det(\mathcal{N}_{\{0\}/\mathbb{A}^2_F})\})$$

to each element of the couple above, which gives:

$$\left(\sum_{p\in I} \sum_{q\in\overline{\{p\}}^{(1)}\cap Z_1} \partial^{\pi_{p,q,0}}_{v_{p,q,0}}(\langle\varphi_1^*(\overline{u_{p,q}})\rangle)\eta\,\chi^{\mathrm{odd}}(m_p m_{p,q}) \otimes (\overline{\pi_{p,q,0}}^* \otimes \varphi_1^*(\tau_{p,q})),\right.$$
$$\left.\sum_{p\in I} \sum_{q\in\overline{\{p\}}^{(1)}\cap Z_2} \partial^{\pi_{p,q,0}}_{v_{p,q,0}}(\langle\varphi_2^*(\overline{u_{p,q}})\rangle)\eta\,\chi^{\mathrm{odd}}(m_p m_{p,q}) \otimes (\overline{\pi_{p,q,0}}^* \otimes \varphi_2^*(\tau_{p,q}))\right)$$

where $\partial^{\pi_{p,q,0}}_{v_{p,q,0}}(\langle\varphi_i^*(\overline{u_{p,q}})\rangle)\eta\,\chi^{\mathrm{odd}}(m_p m_{p,q}) \otimes (\overline{\pi_{p,q,0}}^* \otimes \varphi_i^*(\tau_{p,q})) \in K^{\mathrm{MW}}_{-2}(\kappa(0), \det(\mathcal{N}_{\{0\}/\mathbb{A}^2_F}))$.

By Theorem A.11, for every $i \in \{1,2\}$ we have $\partial^{\pi_{p,q,0}}_{v_{p,q,0}}(\langle\varphi_i^*(\overline{u_{p,q}})\rangle) = \langle\overline{u_{p,q,0}}\rangle\eta\,\chi^{\mathrm{odd}}(m_{p,q,0})$ thus the third step gives:

$$\left(\sum_{p\in I} \sum_{q\in\overline{\{p\}}^{(1)}\cap Z_1} \langle\overline{u_{p,q,0}}\rangle\,\eta^2\,\chi^{\mathrm{odd}}(m_p m_{p,q} m_{p,q,0}) \otimes (\overline{\pi_{p,q,0}}^* \otimes \varphi_1^*(\tau_{p,q})),\right.$$
$$\left.\sum_{p\in I} \sum_{q\in\overline{\{p\}}^{(1)}\cap Z_2} \langle\overline{u_{p,q,0}}\rangle\,\eta^2\,\chi^{\mathrm{odd}}(m_p m_{p,q} m_{p,q,0}) \otimes (\overline{\pi_{p,q,0}}^* \otimes \varphi_2^*(\tau_{p,q}))\right)$$

From Definition B.15 and the notations in Subsection 3.2, using the canonical isomorphism $K^{\mathrm{MW}}_{-2}(F) \simeq \mathrm{W}(F)$ (which sends η^2 to 1), the final step gives:

$$\left(\sum_{p\in I} \sum_{q\in\overline{\{p\}}^{(1)}\cap Z_1} \lambda_{p,q,0}\langle\overline{u_{p,q,0}}\rangle\,\chi^{\mathrm{odd}}(m_p m_{p,q} m_{p,q,0}), \sum_{p\in I} \sum_{q\in\overline{\{p\}}^{(1)}\cap Z_2} \lambda_{p,q,0}\langle\overline{u_{p,q,0}}\rangle\,\chi^{\mathrm{odd}}(m_p m_{p,q} m_{p,q,0})\right)$$

□

4. Examples of computations of the quadratic linking degree

In this Section, we compute the quadratic linking class and degree on examples. To do this we use the method given in Section 3. See Subsection 2.1 for notations and Subsection 2.2 for definitions.

Example 4.1. (Hopf) We define the Hopf link over the perfect field F as follows:

- Z_1 is the intersection of the closed subscheme of $\mathbb{A}^4_F = \mathrm{Spec}(F[x,y,z,t])$ of ideal (x,y) and of $X = \mathbb{A}^4_F \setminus \{0\}$;
- $\varphi_1 : \mathbb{A}^2_F \setminus \{0\} \to X$ is the morphism associated to the morphism of F-algebras $F[x,y,z,t] \to F[u,v]$ which maps x,y,z,t to $0,0,u,v$ respectively;
- $\overline{o_1}$ is the orientation class associated to the couple (x,y) (i.e. the class of the isomorphism $o_1 : \nu_{Z_1} \to \mathcal{O}_{Z_1} \otimes \mathcal{O}_{Z_1}$ which maps $\overline{x}^* \wedge \overline{y}^*$ to $1 \otimes 1$);
- Z_2 is the intersection of the closed subscheme of $\mathbb{A}^4_F$ of ideal (z,t) and of X;
- $\varphi_2 : \mathbb{A}^2_F \setminus \{0\} \to X$ is the morphism associated to the morphism of F-algebras $F[x,y,z,t] \to F[u,v]$ which maps x,y,z,t to $u,v,0,0$ respectively;
- $\overline{o_2}$ is the orientation class associated to the couple (z,t) (i.e. the class of the isomorphism $o_2 : \nu_{Z_2} \to \mathcal{O}_{Z_2} \otimes \mathcal{O}_{Z_2}$ which maps $\overline{z}^* \wedge \overline{t}^*$ to $1 \otimes 1$).

In Table 1 we give oriented fundamental cycles of Z_1 and Z_2, Seifert divisors of Z_1 (with orientation o_1) and Z_2 (with orientation o_2) relative to the link, their intersection product and its image by the boundary map $\partial : H^2(X \setminus Z, \underline{K}^{\mathrm{MW}}_2) \to H^1(Z, \underline{K}^{\mathrm{MW}}_0\{\nu_Z\})$, which is the quadratic

Oriented fund. cycles	$\eta \otimes (\overline{x}^* \wedge \overline{y}^*)$		$\eta \otimes (\overline{z}^* \wedge \overline{t}^*)$
Seifert divisors	$\langle x \rangle \otimes \overline{y}^*$		$\langle z \rangle \otimes \overline{t}^*$
Apply intersection product	$\langle xz \rangle \otimes (\overline{t}^* \wedge \overline{y}^*)$		
Quadratic linking class	$-\langle z \rangle \eta \otimes (\overline{t}^* \wedge \overline{x}^* \wedge \overline{y}^*)$	$\oplus$	$\langle x \rangle \eta \otimes (\overline{y}^* \wedge \overline{z}^* \wedge \overline{t}^*)$
Apply $\widetilde{o_1} \oplus \widetilde{o_2}$	$-\langle z \rangle \eta \otimes \overline{t}^*$	$\oplus$	$\langle x \rangle \eta \otimes \overline{y}^*$
Apply $\varphi_1^* \oplus \varphi_2^*$	$-\langle u \rangle \eta \otimes \overline{v}^*$	$\oplus$	$\langle u \rangle \eta \otimes \overline{v}^*$
Apply $\partial \oplus \partial$	$-\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$	$\oplus$	$\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$
Quadratic linking degree	-1	$\oplus$	1

TABLE 1. The Hopf link

linking class. Then we give cycles which represent $(\widetilde{o_1} \oplus \widetilde{o_2})(\mathrm{Qlc}_{\mathscr{L}})$, $(\varphi_1^* \oplus \varphi_2^*)((\widetilde{o_1} \oplus \widetilde{o_2})(\mathrm{Qlc}_{\mathscr{L}}))$, $(\partial \oplus \partial)((\varphi_1^* \oplus \varphi_2^*)((\widetilde{o_1} \oplus \widetilde{o_2})(\mathrm{Qlc}_{\mathscr{L}})))$ and finally we give the quadratic linking degree. The points over which the cycles live are the obvious ones (for instance $\langle x \rangle \otimes \overline{y}^*$ lives over the generic point of the hypersurface of $X \setminus Z$ of equation $y = 0$).

Note that the rank modulo 2 of each component of the quadratic linking degree of the Hopf link is 1. Note that for every positive even integer k, the image by Σ_k of each component of the quadratic linking degree of the Hopf link is 0. Note that if $F = \mathbb{R}$ then the absolute value of each component of the quadratic linking degree of the Hopf link is equal to 1.

Let us now present examples where the intersection of the underlying divisors is not irreducible (and where the invariants of Corollaries 2.14 and 2.15 and of Theorem 2.17 have different values).

Examples 4.2. (Binary links) Let F be a perfect field of characteristic different from 2 and $a \in F^* \setminus \{-1\}$. We define the binary link B_a over F as follows:

- Z_1 is the intersection of the closed subscheme of $\mathbb{A}^4_F$ of ideal $(f_1 := t - ((1+a)x - y)y, g_1 := z - x(x-y))$ and of X;
- $\varphi_1 : \mathbb{A}^2_F \setminus \{0\} \to X$ is the morphism associated to the morphism of F-algebras $F[x,y,z,t] \to F[u,v]$ which maps x, y, z, t to $u, v, ((1+a)u - v)v, u(u-v)$ respectively;
- $\overline{o_1}$ is the orientation class associated to the couple (f_1, g_1);
- Z_2 is the intersection of the closed subscheme of $\mathbb{A}^4_F$ of ideal $(f_2 := t + ((1+a)x - y)y, g_2 := z + x(x-y))$ and of X;
- $\varphi_2 : \mathbb{A}^2_F \setminus \{0\} \to X$ is the morphism associated to the morphism of F-algebras $F[x,y,z,t] \to F[u,v]$ which maps x, y, z, t to $u, v, -((1+a)u - v)v, -u(u-v)$ respectively;
- $\overline{o_2}$ is the orientation class associated to the couple (f_2, g_2).

In Table 2 we give oriented fundamental cycles of Z_1 and Z_2, Seifert divisors of Z_1 (with orientation o_1) and Z_2 (with orientation o_2) relative to the link, their intersection product and its image by the boundary map $\partial : H^2(X \setminus Z, \underline{K}_2^{\mathrm{MW}}) \to H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\nu_Z\})$, which is the quadratic linking class. Then we give cycles which represent $(\widetilde{o_1} \oplus \widetilde{o_2})(\mathrm{Qlc}_{\mathscr{L}})$, $(\varphi_1^* \oplus \varphi_2^*)((\widetilde{o_1} \oplus \widetilde{o_2})(\mathrm{Qlc}_{\mathscr{L}}))$, $(\partial \oplus \partial)((\varphi_1^* \oplus \varphi_2^*)((\widetilde{o_1} \oplus \widetilde{o_2})(\mathrm{Qlc}_{\mathscr{L}})))$ and finally we give the quadratic linking degree. Unless specified (between parentheses after a central dot), the points over which the cycles live are the obvious ones (for instance $\langle f_1 \rangle \otimes \overline{g_1}^*$ lives over the generic point of the hypersurface of $X \setminus Z$ of equation $g_1 = 0$).

To see how one gets from the fifth line in Table 2 to the sixth line in this Table, note that $-\langle f_2 \rangle \eta \otimes \overline{g_2}^* \cdot (x-y) \in H^1(Z_1, \underline{K}_0^{\mathrm{MW}})$ is equal to $-\langle 2((1+a)x-y)y \rangle \eta \otimes \overline{2x(x-y)}^* \cdot (x-y)$ since in Z_1: $t = ((1+a)x-y)y$ and $z = x(x-y)$. Further note that $-\langle 2((1+a)x-y)y \rangle \eta \otimes \overline{2x(x-y)}^* \cdot (x-y) = -\langle ((1+a)x-y)yx \rangle \eta \otimes \overline{x-y}^* \cdot (x-y)$ and that $-\langle ((1+a)x-y)yx \rangle \eta \otimes \overline{x-y}^* \cdot (x-y) = -\langle ax^3 \rangle \eta \otimes \overline{x-y}^* \cdot (x-y) = -\langle ax \rangle \eta \otimes \overline{x-y}^* \cdot (x-y)$. Similarly, $-\langle f_2 \rangle \eta \otimes \overline{g_2}^* \cdot (x) \in H^1(Z_1, \underline{K}_0^{\mathrm{MW}})$ is equal to $-\langle y \rangle \eta \otimes \overline{x}^* \cdot (x)$.

Note that the rank modulo 2 of each component of the quadratic linking degree of the binary link B_a is 0 (hence the invariant presented in Corollary 2.14 distinguishes between the Hopf link and the binary links). Note that the image by Σ_2 of each component of the quadratic linking degree of the binary link B_a is $\langle a \rangle \in \mathrm{W}(F)/(1)$. For instance, if $F = \mathbb{Q}$, Σ_2 distinguishes between

Or. fund. cyc.	$\eta\otimes(\overline{f_1}^*\wedge\overline{g_1}^*)$		$\eta\otimes(\overline{f_2}^*\wedge\overline{g_2}^*)$
Seifert divisors	$\langle f_1\rangle\otimes\overline{g_1}^*$		$\langle f_2\rangle\otimes\overline{g_2}^*$
Apply inter. prod.	$\langle f_1f_2\rangle\otimes(\overline{g_2}^*\wedge\overline{g_1}^*)\cdot(z,x-y)$ $+\langle f_1f_2\rangle\otimes(\overline{g_2}^*\wedge\overline{g_1}^*)\cdot(z,x)$		
Quad. link. class	$-\langle f_2\rangle\eta\otimes(\overline{g_2}^*\wedge\overline{f_1}^*\wedge\overline{g_1}^*)\cdot(x-y)$ $-\langle f_2\rangle\eta\otimes(\overline{g_2}^*\wedge\overline{f_1}^*\wedge\overline{g_1}^*)\cdot(x)$	$\oplus$	$\langle f_1\rangle\eta\otimes(\overline{g_1}^*\wedge\overline{f_2}^*\wedge\overline{g_2}^*)\cdot(x-y)$ $+\langle f_1\rangle\eta\otimes(\overline{g_1}^*\wedge\overline{f_2}^*\wedge\overline{g_2}^*)\cdot(x)$
Apply $\widetilde{o_1}\oplus\widetilde{o_2}$	$-\langle f_2\rangle\eta\otimes\overline{g_2}^*\cdot(x-y)$ $-\langle f_2\rangle\eta\otimes\overline{g_2}^*\cdot(x)$	$\oplus$	$\langle f_1\rangle\eta\otimes\overline{g_1}^*\cdot(x-y)$ $+\langle f_1\rangle\eta\otimes\overline{g_1}^*\cdot(x)$
Apply $\varphi_1^*\oplus\varphi_2^*$	$-\langle au\rangle\eta\otimes\overline{u-v}^*\cdot(u-v)$ $-\langle v\rangle\eta\otimes\overline{u}^*\cdot(u)$	$\oplus$	$\langle au\rangle\eta\otimes\overline{u-v}^*\cdot(u-v)$ $+\langle v\rangle\eta\otimes\overline{u}^*\cdot(u)$
Apply $\partial\oplus\partial$	$(1+\langle a\rangle)\eta^2\otimes(\overline{u}^*\wedge\overline{v}^*)$	$\oplus$	$-(1+\langle a\rangle)\eta^2\otimes(\overline{u}^*\wedge\overline{v}^*)$
Quad. lk. deg.	$1+\langle a\rangle$	$\oplus$	$-(1+\langle a\rangle)$

TABLE 2. The binary link B_a

all the B_p with p prime numbers since if $p\neq q$ are prime numbers then $\langle p\rangle\in \mathrm{W}(\mathbb{Q})/(1)\leftrightarrow 1\in \mathrm{W}(\mathbb{Z}/p\mathbb{Z})\subset\bigoplus_{r\text{ prime}}\mathrm{W}(\mathbb{Z}/r\mathbb{Z})$ and $\langle q\rangle\in\mathrm{W}(\mathbb{Q})/(1)\leftrightarrow 1\in\mathrm{W}(\mathbb{Z}/q\mathbb{Z})\subset\bigoplus_{r\text{ prime}}\mathrm{W}(\mathbb{Z}/r\mathbb{Z})$ (with the usual isomorphism $\mathrm{W}(\mathbb{Q})/(1)\simeq\bigoplus_{r\text{ prime}}\mathrm{W}(\mathbb{Z}/r\mathbb{Z})$). Note that if $F=\mathbb{R}$ then the absolute value of each component of the quadratic linking degree of the binary link B_a is equal to 2 if $a>0$, to 0 if $a<0$ (hence the invariant presented in Corollary 2.15 distinguishes between the Hopf link and the binary links, as well as between the binary links with positive parameter and the binary links with negative parameter).

The following family of examples is an analogue of the family of torus links $T(2,2n)$ (with $n\geq 1$ an integer) in knot theory. Note that $T(2,2)$ is the Hopf link and that its analogue below is slightly different from the Hopf link in the example above and has quadratic linking degree $(1,-1)$. Note that $T(2,4)$ is the Solomon link.

Examples 4.3 (Torus links)**.** Let $n\geq 1$. Let us define an analogue of the torus link $T(2,2n)$.

Recall that (in knot theory) one of the components of $T(2,2n)$ is the intersection of $\{(a,b)\in\mathbb{C}^2, b=a^n\}$ with $\mathbb{S}^3_\varepsilon$, the 3-sphere of radius ε, and that the other component of $T(2,2n)$ is the intersection of $\{(a,b)\in\mathbb{C}^2, b=-a^n\}$ with $\mathbb{S}^3_\varepsilon$ (for $\varepsilon>0$ small enough). By writing $a=x+iy$ and $b=z+it$ (with $x,y,z,t\in\mathbb{R}$), the equation $b=a^n$ becomes the system of equations

$$\begin{cases} t=\displaystyle\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}(-1)^k x^{n-2k-1}y^{2k+1}\\ z=\displaystyle\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}(-1)^k x^{n-2k}y^{2k}\end{cases}$$

and the equation $b=-a^n$ becomes the system of equations

$$\begin{cases} t=-\displaystyle\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}(-1)^k x^{n-2k-1}y^{2k+1}\\ z=-\displaystyle\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}(-1)^k x^{n-2k}y^{2k}\end{cases}$$

From now on, we denote

$$\Sigma_t(x,y) := \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} (-1)^k x^{n-2k-1} y^{2k+1}, f_1 := t - \Sigma_t(x,y), f_2 := t + \Sigma_t(x,y),$$

$$\Sigma_z(x,y) := \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (-1)^k x^{n-2k} y^{2k}, g_1 := z - \Sigma_z(x,y), g_2 := z + \Sigma_z(x,y)$$

Consequently, we define our analogue over $\mathbb{R}$ of the torus link $T(2,2n)$ as follows:

- Z_1 is the intersection of the closed subscheme of $\mathbb{A}^4_{\mathbb{R}}$ of ideal (f_1, g_1) and of X;
- $\varphi_1 : \mathbb{A}^2_{\mathbb{R}} \backslash \{0\} \to Z_1$ is the morphism associated to the morphism of $\mathbb{R}$-algebras $\mathbb{R}[x,y,z,t] \to \mathbb{R}[u,v]$ which maps x, y, z, t to $u, v, \Sigma_z(u,v), \Sigma_t(u,v)$ respectively;
- $\overline{o_1}$ is the orientation class associated to the couple (f_1, g_1) (i.e. the class of $o_1 : \nu_{Z_1} \to \mathcal{O}_{Z_1} \otimes \mathcal{O}_{Z_1}$ which maps $\overline{f_1}^* \wedge \overline{g_1}^*$ to $1 \otimes 1$);
- Z_2 is the intersection of the closed subscheme of $\mathbb{A}^4_{\mathbb{R}}$ of ideal (f_2, g_2) and of X;
- $\varphi_2 : \mathbb{A}^2_{\mathbb{R}} \backslash \{0\} \to Z_2$ is the morphism associated to the morphism of $\mathbb{R}$-algebras $\mathbb{R}[x,y,z,t] \to \mathbb{R}[u,v]$ which maps x, y, z, t to $u, v, -\Sigma_z(u,v), -\Sigma_t(u,v)$ respectively;
- $\overline{o_2}$ is the orientation class associated to the couple (f_2, g_2) (i.e. the class of $o_2 : \nu_{Z_2} \to \mathcal{O}_{Z_2} \otimes \mathcal{O}_{Z_2}$ which maps $\overline{f_2}^* \wedge \overline{g_2}^*$ to $1 \otimes 1$).

An oriented fundamental cycle of Z_1 (with orientation o_1) is $\eta \otimes (\overline{f_1}^* \wedge \overline{g_1}^*)$ (over the generic point of Z_1) and a Seifert divisor of Z_1 (with orientation o_1) is $\langle f_1 \rangle \otimes \overline{g_1}^*$ (over the generic point of the hypersurface of $X \setminus Z$ of equation $g_1 = 0$).

An oriented fundamental cycle of Z_2 (with orientation o_2) is $\eta \otimes (\overline{f_2}^* \wedge \overline{g_2}^*)$ (over the generic point of Z_2) and a Seifert divisor of Z_2 (with orientation o_2) is $\langle f_2 \rangle \otimes \overline{g_2}^*$ (over the generic point of the hypersurface of $X \setminus Z$ of equation $g_2 = 0$).

The intersection of the underlying divisors has n irreducible components, whose generic points are denoted by $P_0, \ldots, P_{n-1}$, where for all $j \in \{0, \ldots, n-1\}$, the component of generic point P_j is given in $X \setminus Z$ by the equations

$$z = 0, x = \tan\left(\frac{(n-1-2j)\pi}{2n}\right) y$$

Indeed, if we denote $x + iy = \rho e^{i\theta}$ with $\rho \in \mathbb{R}_+^*, \theta \in \mathbb{R}$ then:

$$\begin{aligned} \Re((x+iy)^n) = 0 &\Leftrightarrow \cos(n\theta) = 0 \\ &\Leftrightarrow \theta = \frac{(2j+1)\pi}{2n} \text{ for some } j \in \{0, \ldots, 2n-1\} \\ &\Leftrightarrow x = \tan\left(\frac{(n-1-2j)\pi}{2n}\right) y \text{ for some } j \in \{0, \ldots, n-1\} \end{aligned}$$

From now on, for every $j \in \{0, \ldots, n-1\}$, we denote $\theta_j := \frac{(n-1-2j)\pi}{2n}$. Thus, the homogeneous polynomial $\Sigma_z(x,y)$ of degree n is equal to $\prod_{j=0}^{n-1} (x - \tan(\theta_j) y)$. Note that the $\tan(\theta_j)$, with $j \in \{0, \ldots, n-1\}$, are distinct, since they are the roots of the polynomial $(x+i)^n + (x-i)^n$ (which is coprime with its derivative).

It follows (see Section 3) that the intersection product of these Seifert divisors is equal to:

$$\sum_{j=0}^{n-1} (m_j)_\epsilon \langle f_1 f_2 u_j \rangle \otimes (\overline{\pi_j}^* \wedge \overline{g_1}^*) \cdot (P_j)$$

where π_j (resp. u_j) is a uniformizing parameter (resp. a unit) in $\mathcal{O}_{X \setminus Z, P_j}/(g_1)$ and $m_j \in \mathbb{Z}$ such that $g_2 = u_j \pi_j^{m_j}$. Note that one can choose $\pi_j = g_2$ (hence $m_j = 1$ and $u_j = 1$) since

$\mathcal{O}_{X\setminus Z,P_j}/(g_1) \simeq (\mathbb{R}[x,y,z,t]/(z-\prod_{i=0}^{n-1}(x-\tan(\theta_i)y)))_{(z,x-\tan(\theta_j)y)} \simeq \mathbb{R}[x,y,t]_{(x-\tan(\theta_j)y)}$ and in this ring $g_2 = 2\prod_{i=0}^{n-1}(x-\tan(\theta_i)y)$, thus the intersection product of these Seifert divisors is equal to:

$$\sum_{j=0}^{n-1}\langle f_1 f_2\rangle \otimes (\overline{g_2}^* \wedge \overline{g_1}^*)\cdot(P_j)$$

It follows (see Section 3) that its image by the boundary map, which is the quadratic linking class, is the following:

$$\sum_{j=0}^{n-1} -\langle f_2\rangle\eta \otimes (\overline{g_2}^* \wedge \overline{f_1}^* \wedge \overline{g_1}^*)\cdot(x=\tan(\theta_j)y \text{ in } Z_1)$$
$$+\sum_{j=0}^{n-1}\langle f_1\rangle\eta \otimes (\overline{g_1}^* \wedge \overline{f_2}^* \wedge \overline{g_2}^*)\cdot(x=\tan(\theta_j)y \text{ in } Z_2)$$

Its image by $\widetilde{o_1}\oplus\widetilde{o_2}$ is:

$$\sum_{j=0}^{n-1} -\langle f_2\rangle\eta \otimes \overline{g_2}^* \oplus \sum_{j=0}^{n-1}\langle f_1\rangle\eta\otimes\overline{g_1}^*$$

Its image by $\varphi_1^*\oplus\varphi_2^*$ is:

$$\sum_{j=0}^{n-1} -\langle 2\Sigma_t(u,v)\rangle\eta\otimes\overline{2\Sigma_z(u,v)}^* \oplus \sum_{j=0}^{n-1}\langle -2\Sigma_t(u,v)\rangle\eta\otimes\overline{-2\Sigma_z(u,v)}^*$$

Note that the first component of the couple above is equal to:

$$\sum_{j=0}^{n-1}\langle -\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}(-1)^k(\tan(\theta_j))^{n-2k-1}\prod_{i\neq j,i=0}^{n-1}(\tan(\theta_j)-\tan(\theta_i))v\rangle\eta\otimes\overline{u-\tan(\theta_j)v}^*$$

Its image by the boundary map ∂ is the following:

$$\sum_{j=0}^{n-1}\langle -\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}(-1)^k(\tan(\theta_j))^{n-2k-1}\prod_{i\neq j,i=0}^{n-1}(\tan(\theta_j)-\tan(\theta_i))\rangle\eta^2\otimes(\overline{v}^*\wedge\overline{u-\tan(\theta_j)v}^*)$$
$$=\sum_{j=0}^{n-1}\langle \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}(-1)^k(\tan(\theta_j))^{n-2k-1}\prod_{i\neq j,i=0}^{n-1}(\tan(\theta_j)-\tan(\theta_i))\rangle\eta^2\otimes(\overline{u}^*\wedge\overline{v}^*)$$

Note that $\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}(-1)^k(\tan(\theta_j))^{n-2k-1} = \Im((\tan(\theta_j)+i)^n) = \rho_j\sin(\frac{(2j+1)\pi}{2})$ with ρ_j a positive real number, hence:

$$\langle\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}(-1)^k(\tan(\theta_j))^{n-2k-1}\rangle = \begin{cases}\langle 1\rangle & \text{if } j \text{ is even}\\ \langle -1\rangle & \text{if } j \text{ is odd}\end{cases}$$

Note that for all $l\in\{0,\dots,n-1\}$, $-\frac{\pi}{2}<\theta_l<\frac{\pi}{2}$ hence for all $i<j$, $\tan(\theta_j)-\tan(\theta_i)<0$ and for all $i>j$, $\tan(\theta_j)-\tan(\theta_i)>0$, hence:

$$\langle\prod_{i\neq j,i=0}^{n-1}(\tan(\theta_j)-\tan(\theta_i))\rangle = \begin{cases}\langle 1\rangle & \text{if } j \text{ is even}\\ \langle -1\rangle & \text{if } j \text{ is odd}\end{cases}$$

Therefore $\partial(\varphi_1^*(\widetilde{o_1}(\sigma_{1,\mathscr{L}}))) = n\,\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$, hence the first component of the quadratic linking degree is equal to $n \in \mathrm{W}(\mathbb{R})$.

With similar computations to the ones above, we find that the second component of the quadratic linking degree is equal to $-n \in \mathrm{W}(\mathbb{R})$, hence the quadratic linking degree is equal to $(n, -n) \in \mathrm{W}(\mathbb{R}) \oplus \mathrm{W}(\mathbb{R}) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Note that the rank modulo 2 of each component of the quadratic linking degree of the analogue of $T(2, 2n)$ is 1 if n is odd, 0 if n is even. Note that the absolute value of each component of the quadratic linking degree of the analogue of $T(2, 2n)$ is equal to n (hence the invariant presented in Corollary 2.15 distinguishes between all these links $T(2, 2n)$, similarly to the absolute value of the linking number which distinguishes between all the links $T(2, 2n)$ in knot theory).

Appendix A. An explicit definition of the residue morphisms of Milnor-Witt K-theory

In this Appendix, we give an explicit definition (i.e. one which allows computations) of the noncanonical residue morphism and prove that it is indeed the noncanonical residue morphism (as defined by Morel in [Mor12] and recalled in Definition A.2). Note that explicit definitions of the canonical residue morphism (see Definition A.7) and of the twisted canonical residue morphism (see Definition A.8) follow directly. We use the case $n \leq 0$ in Theorem A.11 to compute the quadratic linking class and degree in Sections 3 and 4; the case $n \geq 1$ is only included for its possible usefulness in other computations.

See [Mor12, Section 3.1] for recollections about Milnor-Witt K-theory. Throughout this Appendix, F is a perfect field, $v : F^* \to \mathbb{Z}$ is a discrete valuation (of residue field $\kappa(v)$ and ring $\mathcal{O}_v$) and π is a uniformizing parameter for v. For all $u \in \mathcal{O}_v^*$, we denote by $\overline{u}$ its class in $\kappa(v)$ (which is in $\kappa(v)^*$ since $u \in \mathcal{O}_v^*$). We denote the usual generators of the Milnor-Witt K-theory ring of F by $[a] \in K_1^{\mathrm{MW}}(F)$ (with $a \in F^*$) and $\eta \in K_{-1}^{\mathrm{MW}}(F)$ (see [Mor12, Definition 3.1]). We denote $\langle a \rangle := 1 + \eta[a] \in K_0^{\mathrm{MW}}(F)$.

Notation A.1. We denote $\epsilon := -\langle -1 \rangle$ and for all $n \in \mathbb{N}_0$, $n_\epsilon := \sum_{i=1}^{n} \langle (-1)^{i-1} \rangle$ and $(-n)_\epsilon := \epsilon\, n_\epsilon$.

We denote by $\chi^{\mathrm{odd}} : \mathbb{Z} \to \{0, 1\}$ the characteristic function of the set of odd numbers.

We now recall Morel's definition of the noncanonical residue morphism.

Definition A.2 (The noncanonical residue morphism)**.** The residue morphism $\partial_v^\pi : K_*^{\mathrm{MW}}(F) \to K_{*-1}^{\mathrm{MW}}(\kappa(v))$ is the only morphism of graded groups which commutes to product by η and satisfies, for all $n \in \mathbb{N}_0, u_1, \ldots, u_n \in \mathcal{O}_v^*$:

$$\partial_v^\pi([\pi, u_1, \ldots, u_n]) = [\overline{u_1}, \ldots, \overline{u_n}] \text{ and } \partial_v^\pi([u_1, \ldots, u_n]) = 0.$$

In [Mor12, Theorem 3.15], Morel proves that such a morphism exists and that it is unique.

Before we define the canonical residue morphism, we recall the following facts and definition:

Proposition A.3 (Proposition 3.17 in [Mor12])**.**

$$\forall u \in \mathcal{O}_v^*, \forall \alpha \in K_*^{\mathrm{MW}}(F),\ \partial_v^\pi(\langle u \rangle \alpha) = \langle \overline{u} \rangle \partial_v^\pi(\alpha)$$

Corollary A.4. *If $\pi' = u'\pi$ with $u' \in \mathcal{O}_v^*$ then $\partial_v^\pi = \langle \overline{u'} \rangle \partial_v^{\pi'}$.*

Definition A.5 (Twisted Milnor-Witt K-theory)**.** Let $m \in \mathbb{Z}$ and L be an F-vector space of dimension 1. The L-twisted m-th Milnor-Witt K-theory abelian group of F, denoted $K_m^{\mathrm{MW}}(F, L)$, is the tensor product of the $\mathbb{Z}[F^*]$-modules $K_m^{\mathrm{MW}}(F)$ and $\mathbb{Z}[L \setminus \{0\}]$ (the scalar product of $K_m^{\mathrm{MW}}(F)$ being $(\sum_{f \in F^*} n_f \lambda_f) \cdot \alpha = \sum_{f \in F^*} n_f \langle f \rangle \alpha$).

Remark A.6. Note that if we fix an isomorphism between L and F then we get an isomorphism of $\mathbb{Z}[F^*]$-modules between $K_m^{\mathrm{MW}}(F, L)$ and $K_m^{\mathrm{MW}}(F)$; nevertheless, $K_m^{\mathrm{MW}}(F, L)$ is a useful construction because there is no canonical isomorphism between L and F (hence no canonical isomorphism between $K_m^{\mathrm{MW}}(F, L)$ and $K_m^{\mathrm{MW}}(F)$, unless $L = F$) and the introduction of $K_m^{\mathrm{MW}}(F, L)$ is what allows us to have canonical residue morphisms.

Definition A.7 (The canonical residue morphism)**.** The canonical residue morphism $\partial_v : K_*^{\mathrm{MW}}(F) \to K_{*-1}^{\mathrm{MW}}(\kappa(v), (\mathfrak{m}_v/\mathfrak{m}_v^2)^\vee)$ (where $\vee$ denotes the dual) is given by $\partial_v = \partial_v^\pi \otimes \overline{\pi}^*$ (with $\overline{\pi}$ the class of π in $\mathfrak{m}_v/\mathfrak{m}_v^2$ (which is nonzero since π is a uniformizing parameter for v) and $\overline{\pi}^*$ its dual basis).

Note that ∂_v does not depend on the choice of π, since if π' is another uniformizing parameter for v then there exists $u' \in \mathcal{O}_v$ such that $\pi' = u'\pi$ hence, by Corollary A.4 , $\partial_v^\pi \otimes \overline{\pi}^* = \langle\overline{u'}\rangle\partial_v^{\pi'} \otimes \overline{\pi}^* = \partial_v^{\pi'} \otimes \overline{u'\pi}^* = \partial_v^{\pi'} \otimes \overline{\pi'}^*$.

Definition A.8 (The twisted canonical residue morphism)**.** Let L be a one-dimensional $\mathcal{O}_v$-module. The twisted canonical residue morphism

$$\partial_{v,L} : K_*^{\mathrm{MW}}(F, L \otimes F) \to K_{*-1}^{\mathrm{MW}}(\kappa(v), (\mathfrak{m}_v/\mathfrak{m}_v^2)^\vee \otimes_{\kappa(v)} (L \otimes \kappa(v)))$$

is given by

$$\partial_{v,L}(\sum_i \alpha_i \otimes l_i) = \sum_i \partial_v^\pi(\alpha_i) \otimes (\overline{\pi}^* \otimes l_i)$$

Before we prove Theorem A.11, we recall the following lemmas.

Lemma A.9. *For all $m, n \in \mathbb{Z}$, $(mn)_\epsilon = m_\epsilon n_\epsilon$.*

Lemma A.10. *For all $m \in \mathbb{Z}$, $\eta m_\epsilon = \eta\chi^{\mathrm{odd}}(m)$.*

Recall that by [Mor12, Lemma 3.6], for all $n \leq 0$, $K_n^{\mathrm{MW}}(F)$ is generated by elements of the form $\langle\pi^m u\rangle\eta^{-n}$ with $m \in \mathbb{Z}$ and $u \in \mathcal{O}_v^*$, hence, since $\partial_v^\pi : K_n^{\mathrm{MW}}(F) \to K_{n-1}^{\mathrm{MW}}(\kappa(v))$ is a group morphism (see Definition A.2), we only need to give $\partial_v^\pi(\langle\pi^m u\rangle\eta^{-n})$.

Recall that by [Mor12, Lemma 3.6], for all $n \geq 1$, $K_n^{\mathrm{MW}}(F)$ is generated by elements of the form $[\pi^{m_1}u_1, \ldots, \pi^{m_n}u_n]$ with $m_1, \ldots, m_n \in \mathbb{Z}$ and $u_1, \ldots, u_n \in \mathcal{O}_v^*$, hence, since $\partial_v^\pi : K_n^{\mathrm{MW}}(F) \to K_{n-1}^{\mathrm{MW}}(\kappa(v))$ is a group morphism (see Definition A.2), we only need to give $\partial_v^\pi([\pi^{m_1}u_1, \ldots, \pi^{m_n}u_n])$.

Theorem A.11. *For all $n \leq 0$, $m \in \mathbb{Z}$ and $u \in \mathcal{O}_v^*$:*

$$\partial_v^\pi(\langle\pi^m u\rangle\eta^{-n}) = \langle\overline{u}\rangle\eta^{-n+1}\chi^{\mathrm{odd}}(m)$$

For all $n \geq 1$, $m_1, \ldots, m_n \in \mathbb{Z}$ and $u_1, \ldots, u_n \in \mathcal{O}_v^*$:

$$\partial_v^\pi([\pi^{m_1}u_1, \ldots, \pi^{m_n}u_n]) =$$

$$\sum_{l=0}^{n-1} \sum_{\substack{J\subset\{1,\ldots,n\},|J|=l \\ J=\{j_1<\cdots<j_l\}}} ((-1)^{\sum_{i=1}^{l} n-l+i-j_i} \prod_{k\in\{1,\ldots,n\}\setminus J} m_k)_\epsilon[\underbrace{-1,\ldots,-1}_{n-1-l \text{ terms}}, \overline{u_{j_1}}, \ldots, \overline{u_{j_l}}]$$

$$+ \sum_{p=1}^{n}\sum_{l=p}^{n} \sum_{\substack{J\subset\{1,\ldots,n\},|J|=l \\ J=\{j_1<\cdots<j_l\}}} (\sum_{I\subset\{1,\ldots,l\},|I|=p} \eta^p\chi^{\mathrm{odd}}(\prod_{i\in I} m_{j_i} \times \prod_{k\in\{1,\ldots,n\}\setminus J} m_k))[\underbrace{-1,\ldots,-1}_{n-1+p-l \text{ terms}}, \overline{u_{j_1}}, \ldots, \overline{u_{j_l}}]$$

Remark A.12. This last formula may seem daunting, but for $n = 1$ it is merely

$$\partial_v^\pi([\pi^m u]) = m_\epsilon + \eta\chi^{\mathrm{odd}}(m)[\overline{u}]$$

(i.e. $\partial_v^\pi([\pi^m u]) = \langle\overline{u}\rangle m_\epsilon$, similarly to the case $n \leq 0$ where $\partial_v^\pi(\langle\pi^m u\rangle\eta^{-n}) = \langle\overline{u}\rangle\eta^{-n+1}m_\epsilon$, see Lemma A.10), for $n = 2$ it is merely

$$\begin{aligned}\partial_v^\pi([\pi^{m_1}u_1, \pi^{m_2}u_2]) = {} & (m_1m_2)_\epsilon[-1] + (-m_2)_\epsilon[\overline{u_1}] + (m_1)_\epsilon[\overline{u_2}] \\ & + \eta\chi^{\mathrm{odd}}(m_1m_2)[-1, \overline{u_1}] + \eta\chi^{\mathrm{odd}}(m_1m_2)[-1, \overline{u_2}] \\ & + (\eta\chi^{\mathrm{odd}}(m_1) + \eta\chi^{\mathrm{odd}}(m_2))[\overline{u_1}, \overline{u_2}] \\ & + \eta^2\chi^{\mathrm{odd}}(m_1m_2)[-1, \overline{u_1}, \overline{u_2}]\end{aligned}$$

and so on.

Proof. Let $n \leq 0$, $m \in \mathbb{Z}$ and $u \in \mathcal{O}_v^*$.

$$\begin{aligned}
\partial_v^\pi(\langle \pi^m u\rangle \eta^{-n}) &= \partial_v^\pi((1+\eta[\pi^m u])\eta^{-n}) \\
&= \partial_v^\pi((1+\eta([\pi^m]+[u]+\eta[\pi^m,u]))\eta^{-n}) \\
&= \partial_v^\pi((1+\eta m_\epsilon[\pi]+\eta[u]+\eta^2 m_\epsilon[\pi,u])\eta^{-n}) && \text{by [Mor12, Lemma 3.14]} \\
&= \eta^{-n}\partial_v^\pi(1)+\eta^{-n+1}m_\epsilon\partial_v^\pi([\pi]) \\
&\quad +\eta^{-n+1}\partial_v^\pi([u])+\eta^{-n+2}m_\epsilon\partial_v^\pi([\pi,u]) && \text{by Prop. A.3 and Def. A.2} \\
&= \eta^{-n+1}m_\epsilon+\eta^{-n+2}m_\epsilon[\overline{u}] && \text{by Def. A.2} \\
&= (\eta^{-n+1}+\eta^{-n+2}[\overline{u}])\chi^{\mathrm{odd}}(m) && \text{by Lemma A.10} \\
&= \langle\overline{u}\rangle\eta^{-n+1}\chi^{\mathrm{odd}}(m)
\end{aligned}$$

Let $n \geq 1, m_1, \dots, m_n \in \mathbb{Z}$, $u_1, \dots, u_n \in \mathcal{O}_v^*$ and $N := \{1, \dots, n\}$.

$$\begin{aligned}
[\pi^{m_1}u_1,\dots,\pi^{m_n}u_n] &= \prod_{i=1}^n([\pi^{m_i}]+[u_i]+\eta[\pi^{m_i},u_i]) \\
&= \prod_{i=1}^n((m_i)_\epsilon[\pi]+[u_i]+\eta(m_i)_\epsilon[\pi,u_i]) \text{ by [Mor12, Lemma 3.14]}
\end{aligned}$$

Hence $[\pi^{m_1}u_1,\dots,\pi^{m_n}u_n] = \sum_{l=0}^n \sum_{\substack{J\subset\{1,\dots,n\},|J|=l\\ J=\{j_1<\dots<j_l\}}} \prod_{k\in N\setminus J}(m_k)_\epsilon \times \epsilon^{\sum_{i=1}^l n-l+i-j_i}[\pi,\dots,\pi,u_{j_1},\dots,u_{j_l}]$

$$+\sum_{p=1}^n\sum_{l=p}^n \sum_{\substack{J\subset\{1,\dots,n\},|J|=l\\ J=\{j_1<\dots<j_l\}}} \Big(\sum_{I\subset\{1,\dots,l\},|I|=p} \eta^p \times \prod_{i\in I}(m_{j_i})_\epsilon \times \prod_{k\in N\setminus J}(m_k)_\epsilon\Big)[\pi,\dots,\pi,u_{j_1},\dots,u_{j_l}]$$

We obtained this last equality by developing the product and using [Mor12, Corollary 3.8] (ϵ-graded commutativity), as well as the fact that $\eta\epsilon = \eta$.

The index p corresponds to the number of terms coming from an $\eta(m_i)_\epsilon[\pi, u_i]$, the index l corresponds to the number of terms coming from a $[u_i]$ or an $\eta(m_i)_\epsilon[\pi, u_i]$, the set $J = \{j_1, \dots, j_l\}$ (with $j_1 < \dots < j_l$) corresponds to the indices of the terms coming from a $[u_i]$ or an $\eta(m_i)_\epsilon[\pi, u_i]$ and the set I corresponds to the indices of the j_i such that u_{j_i} comes from an $\eta(m_{j_i})_\epsilon[\pi, u_{j_i}]$.

By [Mor12, Lemma 3.7] and Lemma A.9:

$$[\pi^{m_1}u_1,\dots,\pi^{m_n}u_n] = \sum_{l=0}^n \sum_{\substack{J\subset\{1,\dots,n\},|J|=l\\ J=\{j_1<\dots<j_l\}}} ((-1)^{\sum_{i=1}^l n-l+i-j_i}\prod_{k\in N\setminus J} m_k)_\epsilon[\pi,-1,\dots,-1,u_{j_1},\dots,u_{j_l}]+$$

$$\sum_{p=1}^n\sum_{l=p}^n \sum_{\substack{J\subset\{1,\dots,n\},|J|=l\\ J=\{j_1<\dots<j_l\}}} \Big(\sum_{I\subset\{1,\dots,l\},|I|=p} \eta^p(\prod_{i\in I}m_{j_i} \times \prod_{k\in N\setminus J} m_k)_\epsilon\Big)[\pi,-1,\dots,-1,u_{j_1},\dots,u_{j_l}]$$

By Lemma A.10 :

$$[\pi^{m_1}u_1,\dots,\pi^{m_n}u_n] = \sum_{l=0}^n \sum_{\substack{J\subset\{1,\dots,n\},|J|=l\\ J=\{j_1<\dots<j_l\}}} ((-1)^{\sum_{i=1}^l n-l+i-j_i}\prod_{k\in N\setminus J} m_k)_\epsilon[\pi,-1,\dots,-1,u_{j_1},\dots,u_{j_l}]+$$

$$\sum_{p=1}^n\sum_{l=p}^n \sum_{\substack{J\subset\{1,\dots,n\},|J|=l\\ J=\{j_1<\dots<j_l\}}} \Big(\sum_{I\subset\{1,\dots,l\},|I|=p} \eta^p\chi^{\mathrm{odd}}(\prod_{i\in I}m_{j_i} \times \prod_{k\in N\setminus J} m_k)\Big)[\pi,-1,\dots,-1,u_{j_1},\dots,u_{j_l}]$$

By Def. A.2 and Prop. A.3 , $\partial_v^\pi([\pi^{m_1}u_1,\dots,\pi^{m_n}u_n])$ is equal to:

$$\sum_{l=0}^{n-1}\sum_{\substack{J\subset\{1,\dots,n\},|J|=l\\ J=\{j_1<\dots<j_l\}}}((-1)^{\sum_{i=1}^{l} n-l+i-j_i}\prod_{k\in N\setminus J} m_k)_\epsilon[-1,\dots,-1,u_{j_1},\dots,u_{j_l}]+$$

$$\sum_{p=1}^{n}\sum_{l=p}^{n}\sum_{\substack{J\subset\{1,\dots,n\},|J|=l\\ J=\{j_1<\dots<j_l\}}}(\sum_{I\subset\{1,\dots,l\},|I|=p}\eta^p\chi^{\mathrm{odd}}(\prod_{i\in I} m_{j_i}\times\prod_{k\in N\setminus J} m_k))[-1,\dots,-1,u_{j_1},\dots,u_{j_l}]$$

Note that the term $l=n$ in the first double sum vanishes since $\partial_v^\pi([u_1,\dots,u_n])=0$ (by Definition A.2). □

Appendix B. The Rost-Schmid complex and Rost-Schmid groups

In this Appendix, we recall notions about the Rost-Schmid complex and its cohomology groups which are used in our paper. See [Mor12, Section 3.1] for recollections about Milnor-Witt K-theory. Throughout this Appendix, F is a perfect field and X is a smooth F-scheme. We denote the usual generators of the Milnor-Witt K-theory ring of F by $[a]\in K_1^{\mathrm{MW}}(F)$ (with $a\in F^*$) and $\eta\in K_{-1}^{\mathrm{MW}}(F)$ (see [Mor12, Definition 3.1]). We denote $\langle a\rangle := 1+\eta[a]\in K_0^{\mathrm{MW}}(F)$.

B.1. Definitions and first properties.

The Rost-Schmid complex was introduced by Morel in [Mor12, Chapter 5]. Before we define it, recall Definition A.5 (twisted Milnor-Witt K-theory) and the following Definition.

Definition B.1 (Determinant of a locally free module)**.** The determinant of a locally free $\mathcal{O}_X$-module $\mathcal{V}$ of constant finite rank r, denoted $\det(\mathcal{V})$, is its r-th exterior power $\Lambda^r(\mathcal{V})$.

Definition B.2 (Rost-Schmid complex)**.** Let $j\in\mathbb{Z}$ and $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The Rost-Schmid complex associated to X, j and $\mathcal{L}$ is :

$$\mathcal{C}(X,\underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\})=\bigoplus_{i\in\mathbb{N}_0}\mathcal{C}^i(X,\underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\})$$

with

$$\mathcal{C}^i(X,\underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\})=\bigoplus_{x\in X^{(i)}}K_{j-i}^{\mathrm{MW}}(\kappa(x),\nu_x\otimes_{\kappa(x)}\mathcal{L}_{|x})$$

where $\mathcal{L}_{|x}=\mathcal{L}_x\otimes_{\mathcal{O}_{X,x}}\kappa(x)$ and $\nu_x=\det(\mathcal{N}_{x/X})$ with $\mathcal{N}_{x/X}$ the normal sheaf of x in X, i.e. the dual of $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$. We denote $\mathcal{C}(X,\underline{K}_j^{\mathrm{MW}}):=\mathcal{C}(X,\underline{K}_j^{\mathrm{MW}}\{\mathcal{O}_X\})$.

Let us now introduce the differential of the complex in a special case.

Notation B.3. Let X be a smooth integral F-scheme of generic point x. Let $y\in X^{(1)}$ and $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We denote by

$$\partial_y^x: K_*^{\mathrm{MW}}(\kappa(x),\nu_x\otimes_{\kappa(x)}\mathcal{L}_{|x})\to K_{*-1}^{\mathrm{MW}}(\kappa(y),\nu_y\otimes_{\kappa(y)}\mathcal{L}_{|y})$$

the twisted canonical residue morphism associated to the discrete valuation of $\mathcal{O}_{X,y}$ (see Definition A.8).

In [Mor12, pp. 121-122] (just before Definition 5.11), Morel defines morphisms

$$\partial_y^x: K_*^{\mathrm{MW}}(\kappa(x),\nu_x\otimes_{\kappa(x)}\mathcal{L}_{|x})\to K_{*-1}^{\mathrm{MW}}(\kappa(y),\nu_y\otimes_{\kappa(y)}\mathcal{L}_{|y})$$

where X is a smooth F-scheme, $x\in X$ and $y\in\overline{\{x\}}^{(1)}$. See also Déglise's notes [Dég22] and Feld's article [Fel20] (take $M=\underline{K}^{\mathrm{MW}}$ in Feld's notations). In the case where X is integral of generic point x, we get the morphism in Notation B.3.

Definition B.4 (Differential of the Rost-Schmid complex)**.** Let X be a smooth F-scheme, $j \in \mathbb{Z}$ and $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The differential of the Rost-Schmid complex associated to X, j and $\mathcal{L}$ is the morphism $d_{X,j,\mathcal{L}} : \mathcal{C}^*(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\}) \to \mathcal{C}^{*+1}(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\})$, denoted d for short, given by $d^i(\sum_{x \in X^{(i)}} k_x) = \sum_{x \in X^{(i)}} \sum_{y \in \overline{\{x\}}^{(1)}} \partial_y^x(k_x)$.

Note that the sum which appears in the above definition is well-defined (since, with the same notations as above, for every k_x the number of $y \in \overline{\{x\}}^{(1)}$ such that $\partial_y^x(k_x) \neq 0$ is finite (see Déglise's notes [Dég22])).

By [Mor12, Theorem 5.31], the Rost-Schmid complex is a complex, i.e. for all $i \in \mathbb{N}_0$, $d^{i+1} \circ d^i = 0$, hence we can define the Rost-Schmid groups as follows.

Definition B.5 (Rost-Schmid groups)**.** Let $i, j \in \mathbb{Z}$, $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The i-th Rost-Schmid group associated to X, j and $\mathcal{L}$, denoted by $H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\})$, is the i-th cohomology group of the Rost-Schmid complex $\mathcal{C}(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\})$, i.e.:

$$H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\}) = \ker(d^i)/\operatorname{im}(d^{i-1})$$

where by convention $d^i = 0$ if $i < 0$. We denote $H^i(X, \underline{K}_j^{\mathrm{MW}}) := H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{O}_X\})$.

Note that by [Mor12, Theorem 5.47] Rost-Schmid groups generalize Chow-Witt groups $\widetilde{CH}^i(X)$: if X is a smooth F-scheme and $i \in \mathbb{N}_0$ then $H^i(X, \underline{K}_i^{\mathrm{MW}}) = \widetilde{CH}^i(X)$.

Let us now state the property of homotopy invariance of Rost-Schmid groups.

Theorem B.6 (Theorem 5.38 in [Mor12])**.** *Let $\pi : \mathbb{A}_X^1 \to X$ be the projection, $i \in \mathbb{N}_0$ and $j \in \mathbb{Z}$. The induced morphism $\pi^* : H^i(X, \underline{K}_j^{\mathrm{MW}}) \to H^i(\mathbb{A}_X^1, \underline{K}_j^{\mathrm{MW}})$ is an isomorphism.*

We now define boundary triples and boundary maps, which were introduced by Feld in [Fel20] (following what Rost did in [Ros96]).

Definition B.7 (Boundary triple)**.** A boundary triple is a 5-tuple (Z, i, X, j, U), or abusively a triple (Z, X, U), with $i : Z \to X$ a closed immersion and $j : U \to X$ an open immersion such that the image of U by j is the complement in X of the image of Z by i, where Z, X, U are smooth F-schemes of pure dimensions. We denote by d_Z and d_X the dimensions of Z and X respectively and by ν_Z the determinant of the normal sheaf of Z in X.

Notation B.8. Let (Z, i, X, j, U) be a boundary triple. We have a canonical isomorphism $\mathcal{C}^\bullet(X, \underline{K}_*^{\mathrm{MW}}) \simeq \mathcal{C}^{\bullet+d_Z-d_X}(Z, \underline{K}_{*+d_Z-d_X}^{\mathrm{MW}}\{\nu_Z\}) \oplus \mathcal{C}^\bullet(U, \underline{K}_*^{\mathrm{MW}})$. We denote the projections by $i^* : \mathcal{C}^\bullet(X, \underline{K}_*^{\mathrm{MW}}) \to \mathcal{C}^{\bullet+d_Z-d_X}(Z, \underline{K}_{*+d_Z-d_X}^{\mathrm{MW}}\{\nu_Z\})$ and $j^* : \mathcal{C}^\bullet(X, \underline{K}_*^{\mathrm{MW}}) \to \mathcal{C}^\bullet(U, \underline{K}_*^{\mathrm{MW}})$ and the inclusions by $i_* : \mathcal{C}^{\bullet+d_Z-d_X}(Z, \underline{K}_{*+d_Z-d_X}^{\mathrm{MW}}\{\nu_Z\}) \to \mathcal{C}^\bullet(X, \underline{K}_*^{\mathrm{MW}})$ and $j_* : \mathcal{C}^\bullet(U, \underline{K}_*^{\mathrm{MW}}) \to \mathcal{C}^\bullet(X, \underline{K}_*^{\mathrm{MW}})$.

Remark B.9. Note that the morphisms i_* and j^* commute with the differentials of the Rost-Schmid complexes and induce morphisms $i_* : H^n(Z, \underline{K}_m^{\mathrm{MW}}\{\nu_Z\}) \to H^{n+d_X-d_Z}(X, \underline{K}_{m+d_X-d_Z}^{\mathrm{MW}})$ (which is also induced by the push-forward of the closed immersion i) and $j^* : H^n(X, \underline{K}_m^{\mathrm{MW}}) \to H^n(U, \underline{K}_m^{\mathrm{MW}})$ (which is also induced by the pull-back of the open immersion j).

Definition B.10 (Boundary map)**.** Let (Z, i, X, j, U) be a boundary triple. The boundary map associated to this boundary triple is the morphism

$$\partial : \mathcal{C}^\bullet(U, \underline{K}_*^{\mathrm{MW}}) \to \mathcal{C}^{\bullet+1+d_Z-d_X}(Z, \underline{K}_{*+d_Z-d_X}^{\mathrm{MW}}\{\nu_Z\})$$

induced by the differential d of the Rost-Schmid complex $\mathcal{C}(X, \underline{K}_*^{\mathrm{MW}})$, i.e.:

$$\partial = i^* \circ d \circ j_*$$

The following Theorem is a special case of the more general exact triangle theorem in homological algebra (the boundary maps being the connecting morphisms).

Theorem B.11. *Let (Z,i,X,j,U) be a boundary triple. The boundary map induces a morphism $\partial : H^{n+d_X-d_Z}(U,\underline{K}^{\mathrm{MW}}_{m+d_X-d_Z}) \to H^{n+1}(Z,\underline{K}^{\mathrm{MW}}_m\{\nu_Z\})$ and we have the following long exact sequence, called the localization long exact sequence:*

$$\dots \longrightarrow H^n(Z,\underline{K}^{\mathrm{MW}}_m\{\nu_Z\}) \xrightarrow{i_*} H^{n+d_X-d_Z}(X,\underline{K}^{\mathrm{MW}}_{m+d_X-d_Z}) \xrightarrow{j^*}$$

$$\xrightarrow{j^*} H^{n+d_X-d_Z}(U,\underline{K}^{\mathrm{MW}}_{m+d_X-d_Z}) \xrightarrow{\partial} H^{n+1}(Z,\underline{K}^{\mathrm{MW}}_m\{\nu_Z\}) \longrightarrow \dots$$

B.2. **The Rost-Schmid groups of punctured affine spaces.** Let us now compute the Rost-Schmid groups of $\mathbb{A}^n_F \setminus \{0\}$ for $n \geq 2$. To do this, we use the following Lemma (which is also used in the main part of the paper).

Lemma B.12. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. For all $i,j \in \mathbb{Z}$, the morphism*

$$\begin{cases} \mathcal{C}^i(X,\underline{K}^{\mathrm{MW}}_j\{\mathcal{L}\otimes\mathcal{L}\}) & \to & \mathcal{C}^i(X,\underline{K}^{\mathrm{MW}}_j) \\ \sum\limits_{x\in I} k_x \otimes (l_x \otimes l_x) & \mapsto & \sum\limits_{x\in I} k_x \end{cases}$$

where I is a finite subset of $X^{(i)}$, $k_x \in K^{\mathrm{MW}}_{j-i}(\kappa(x),\nu_x)$ and $l_x \in \mathcal{L}_x \setminus \{0\}$, is a well-defined isomorphism which commutes with differentials.

Proof. Note that elements of $\mathcal{C}^i(X,\underline{K}^{\mathrm{MW}}_j\{\mathcal{L}\otimes\mathcal{L}\})$ are of the form $\sum_{x\in I} m_x \otimes t_x$ with $m_x \in K^{\mathrm{MW}}_{j-i}(\kappa(x))$ and $t_x \in \mathbb{Z}[(\nu_x \otimes (\mathcal{L}\otimes\mathcal{L})_{|x}) \setminus \{0\}]$. Let $x \in I$. Since $\nu_x \otimes (\mathcal{L}\otimes\mathcal{L})_{|x}$ is a $\kappa(x)$-vector space of dimension 1, there exist $n_x \in K^{\mathrm{MW}}_{j-i}(\kappa(x))$ and $s_x \in (\nu_x \otimes (\mathcal{L}\otimes\mathcal{L})_{|x}) \setminus \{0\}$ such that $m_x \otimes t_x = n_x \otimes s_x$. By definition of $K^{\mathrm{MW}}_{j-i}(\kappa(x),\nu_x)$, there exist $h_x \in K^{\mathrm{MW}}_{j-i}(\kappa(x),\nu_x)$ and $l_x, r_x \in \mathcal{L}_{|x} \setminus \{0\}$ such that $n_x \otimes s_x = h_x \otimes (l_x \otimes r_x)$. Since $\mathcal{L}_{|x}$ is a $\kappa(x)$-vector space of dimension 1, there exists $v_x \in \kappa(x)^*$ such that $r_x = v_x l_x$. It follows that $h_x \otimes (l_x \otimes r_x) = \langle v_x \rangle h_x \otimes (l_x \otimes l_x)$. Denoting $k_x := \langle v_x \rangle h_x$, we get that elements of $\mathcal{C}^i(X,\underline{K}^{\mathrm{MW}}_j\{\mathcal{L}\otimes\mathcal{L}\})$ are of the form $\sum_{x\in I} k_x \otimes (l_x \otimes l_x)$. This morphism is well-defined since if $\sum_{x\in I} k_x \otimes (l_x \otimes l_x) = \sum_{x\in J} k'_x \otimes (l'_x \otimes l'_x)$ then for all x which are not in $I \cap J$, $k_x = k'_x = 0$, and for all $x \in I \cap J$, $l'_x = u_x l_x$ for some $u_x \in F^*$ and $k'_x \otimes (l'_x \otimes l'_x) = \langle u_x^2 \rangle k'_x \otimes (l_x \otimes l_x) = k'_x \otimes (l_x \otimes l_x)$ hence $k'_x \otimes (l_x \otimes l_x) = k_x \otimes (l_x \otimes l_x)$ hence $k'_x = k_x$. The preceding equality $k'_x \otimes (l'_x \otimes l'_x) = k'_x \otimes (l_x \otimes l_x)$ shows that the morphism

$$\begin{cases} \mathcal{C}^i(X,\underline{K}^{\mathrm{MW}}_j) & \to & \mathcal{C}^i(X,\underline{K}^{\mathrm{MW}}_j\{\mathcal{L}\otimes\mathcal{L}\}) \\ \sum\limits_{x\in I} k_x & \mapsto & \sum\limits_{x\in I} k_x \otimes (l_x \otimes l_x) \end{cases}$$

is well-defined, which shows that the morphism is an isomorphism as announced. It is straightforward to show that it commutes with differentials. □

Proposition B.13. *Let $n, i \in \mathbb{N}_0$, $j \in \mathbb{Z}$ with $n \geq 2$. The Rost-Schmid group $H^i(\mathbb{A}^n_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_j)$ is isomorphic to $K^{\mathrm{MW}}_j(F)$ if $i = 0$, to $K^{\mathrm{MW}}_{j-n}(F)$ if $i = n-1$, to 0 otherwise.*

Proof. The heart of the proof consists in using the localization long exact sequence associated to the boundary triple $(\{0\}, \mathbb{A}^n_F, \mathbb{A}^n_F \setminus \{0\})$. We first determine the Rost-Schmid groups of $\{0\}$, i.e. of $\mathrm{Spec}(F)$, and of $\mathbb{A}^n_F$.

By definition, for all $j \in \mathbb{Z}$ and $i > 0$, $\mathcal{C}^i(\mathrm{Spec}(F), \underline{K}^{\mathrm{MW}}_j) = 0$, $\mathcal{C}^0(\mathrm{Spec}(F), \underline{K}^{\mathrm{MW}}_j) = K^{\mathrm{MW}}_j(F,F) = K^{\mathrm{MW}}_j(F)$ and the d^i are zero morphisms, hence $H^i(\mathrm{Spec}(F), \underline{K}^{\mathrm{MW}}_j) = 0$ and $H^0(\mathrm{Spec}(F), \underline{K}^{\mathrm{MW}}_j) = K^{\mathrm{MW}}_j(F)$.

By Theorem B.6 (homotopy invariance), for all $i \in \mathbb{N}_0$ and $j \in \mathbb{Z}$, $H^i(\mathbb{A}^n_F, \underline{K}^{\mathrm{MW}}_j)$ is canonically isomorphic to $H^i(\mathrm{Spec}(F), \underline{K}^{\mathrm{MW}}_j)$ hence to $K^{\mathrm{MW}}_j(F)$ if $i = 0$, to 0 otherwise.

The localization long exact sequence (see Theorem B.11) associated to the boundary triple $(\{0\}, \varphi, \mathbb{A}^n_F, \psi, \mathbb{A}^n_F \setminus \{0\})$ gives us the following exact sequences for all $j \in \mathbb{Z}$ and $i \notin \{0, n-1\}$:

$$0 \longrightarrow H^0(\mathbb{A}^n_F, \underline{K}^{\mathrm{MW}}_j) \simeq K^{\mathrm{MW}}_j(F) \xrightarrow{\psi^*} H^0(\mathbb{A}^n_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_j) \longrightarrow 0$$

$$0 \longrightarrow H^{n-1}(\mathbb{A}^n_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_j) \xrightarrow{\partial} H^0(\{0\}, \underline{K}^{\mathrm{MW}}_{j-n}\{\det(\mathcal{N}_{\{0\}/\mathbb{A}^n_F})\}) \simeq K^{\mathrm{MW}}_{j-n}(F) \longrightarrow 0$$

$$0 \longrightarrow H^i(\mathbb{A}^n_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_j) \longrightarrow 0$$

Thus $H^i(\mathbb{A}^n_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_j)$ is isomorphic to $K^{\mathrm{MW}}_j(F)$ if $i = 0$, to $K^{\mathrm{MW}}_{j-n}(F)$ if $i = n-1$, to 0 otherwise. Note that the isomorphism $H^0(\{0\}, \underline{K}^{\mathrm{MW}}_{j-n}\{\det(\mathcal{N}_{\{0\}/\mathbb{A}^n_F})\}) \simeq K^{\mathrm{MW}}_{j-n}(F)$ is the composite of the isomorphism induced by an isomorphism $\det(\mathcal{N}_{\{0\}/\mathbb{A}^n_F}) \to \mathcal{O}_{\{0\}} \otimes \mathcal{O}_{\{0\}}$ (which depends on the choice of such an isomorphism) and of the isomorphism induced by the one in Lemma B.12. □

Remark B.14. Note that this Proposition is already known (combine [AF14, Lemma 4.5] with [Mor12, Corollary 5.43], [Fel21, Example 1.5.1.19] and [Mor12, Theorem 5.46]), but the proof we did above is important for the following Definition (which is used in the definition of the quadratic linking degree, see Definition 2.11).

Definition B.15 (The conventional isomorphism)**.** The conventional isomophism

$$\zeta : H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_0) \to \mathrm{W}(F)$$

is the composite of the boundary isomorphism

$$\partial : H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_0) \to H^0(\{0\}, \underline{K}^{\mathrm{MW}}_{-2}\{\det(\mathcal{N}_{\{0\}/\mathbb{A}^n_F})\})$$

(see the proof of Proposition B.13), of the isomorphism

$$H^0(\{0\}, \underline{K}^{\mathrm{MW}}_{-2}\{\det(\mathcal{N}_{\{0\}/\mathbb{A}^2_F})\}) \to K^{\mathrm{MW}}_{-2}(F)$$

induced by the isomorphism $\det(\mathcal{N}_{\{0\}/\mathbb{A}^2_F}) \to \mathcal{O}_{\{0\}} \otimes \mathcal{O}_{\{0\}}$ which sends $\overline{u}^* \wedge \overline{v}^*$ to $1 \otimes 1$, where $\mathbb{A}^2_F = \mathrm{Spec}(F[u,v])$ (see the proof of Proposition B.13) and of the canonical isomorphism

$$K^{\mathrm{MW}}_{-2}(F) \to \mathrm{W}(F)$$

(which sends η^2 to 1).

B.3. **The intersection product of oriented divisors.** See Fasel's chapter [Fas20], paragraph 3.4, for the intersection product of oriented divisors in X, i.e. the intersection product $\cdot$: $H^1(X, \underline{K}^{\mathrm{MW}}_1) \times H^1(X, \underline{K}^{\mathrm{MW}}_1) \to H^2(X, \underline{K}^{\mathrm{MW}}_2)$, or, in other words, $\cdot : \widetilde{CH}^1(X) \times \widetilde{CH}^1(X) \to \widetilde{CH}^2(X)$ (take $\mathcal{L} = \mathcal{O}_X$ and $a = 0$ in Fasel's notations). We use this intersection product to define the quadratic linking class (hence also to define the quadratic linking degree, see Definitions 2.8 and 2.11). We also need the following Propositions:

Proposition B.16 (Paragraph 3.4 [Fas20])**.** *The intersection product makes the Chow-Witt ring* $\widetilde{CH}^*(X) = \bigoplus_{i \in \mathbb{N}_0} \widetilde{CH}^i(X)$ *into a graded* $K^{\mathrm{MW}}_0(F)$*-algebra.*

Proposition B.17 (Paragraph 3.4 in [Fas20])**.** *Let c_1, c_2 be oriented divisors in X, i.e. $c_1, c_2 \in H^1(X, \underline{K}^{\mathrm{MW}}_1)$. The intersection product of c_1 with c_2, denoted by $c_1 \cdot c_2$, is $\langle -1 \rangle$-commutative:*

$$c_2 \cdot c_1 = \langle -1 \rangle (c_1 \cdot c_2)$$

We use the following formula (or rather the formula in Corollary B.19), proved by Déglise in [Dég22], to compute the intersection product in our examples (see Sections 3 and 4).

Theorem B.18. *Let X be a smooth F-scheme. Let D_1, D_2 be smooth integral divisors in X such that $D_1 \cap D_2$ is of codimension 2 in X. For all $i \in \{1,2\}$, let f_i be a local parameter for D_i, i.e. f_i is a uniformizing parameter for $\mathcal{O}_{X,D_i}$. The intersection product of $1 \otimes \overline{f_1}^* \in H^1(X, \underline{K}^{\mathrm{MW}}_1)$ (over the generic point of D_1) with $1 \otimes \overline{f_2}^* \in H^1(X, \underline{K}^{\mathrm{MW}}_1)$ (over the generic point of D_2) is the class in $H^2(X, \underline{K}^{\mathrm{MW}}_2)$ of the sum over $\overline{\{x\}}$ irreducible component of $D_1 \cap D_2$ of $((m_x)_\epsilon \langle u_x \rangle) \otimes (\overline{\pi_x}^* \otimes \overline{f_1}^*)$ (over the point x), where π_x is a uniformizing parameter for $\mathcal{O}_{X,x}/(f_1)$, u_x is a unit in $\mathcal{O}_{X,x}/(f_1)$, $m_x \in \mathbb{Z}$ and $f_2 = u_x \pi_x^{m_x} \in \mathcal{O}_{X,x}/(f_1)$.*

Corollary B.19. *Let X be a smooth F-scheme. Let D_1, D_2 be smooth integral divisors in X such that $D_1 \cap D_2$ is of codimension 2 in X. For all $i \in \{1,2\}$, let f_i be a local parameter for D_i and g_i be a unit in $\kappa(D_i) = \mathcal{O}_{X,D_i}/\mathfrak{m}_{X,D_i}$. The intersection product of $\langle g_1 \rangle \otimes \overline{f_1}^* \in H^1(X, \underline{K}_1^{\mathrm{MW}})$ (over the generic point of D_1) with $\langle g_2 \rangle \otimes \overline{f_2}^* \in H^1(X, \underline{K}_1^{\mathrm{MW}})$ (over the generic point of D_2) is the class in $H^2(X, \underline{K}_2^{\mathrm{MW}})$ of the sum over $\overline{\{x\}}$ irreducible component of $D_1 \cap D_2$ of $((m_x)_\epsilon \langle g_1 g_2 u_x \rangle) \otimes (\overline{\pi_x}^* \otimes \overline{f_1}^*)$ (over the point x), where π_x is a uniformizing parameter for $\mathcal{O}_{X,x}/(f_1)$, u_x is a unit in $\mathcal{O}_{X,x}/(f_1)$, $m_x \in \mathbb{Z}$ and $f_2 = u_x \pi_x^{m_x} \in \mathcal{O}_{X,x}/(f_1)$.*

Proof. Note that for all $i \in \{1,2\}$, $\langle g_i \rangle \otimes \overline{f_i}^* = 1 \otimes \overline{g_i f_i}^*$ with $g_i f_i$ a local parameter for D_i ($\overline{g_i f_i} \in \mathfrak{m}_{X,D_i}/\mathfrak{m}_{X,D_i}^2$ is well-defined since $g_i \in \mathcal{O}_{X,D_i}/\mathfrak{m}_{X,D_i}$ and $\overline{f_i} \in \mathfrak{m}_{X,D_i}/\mathfrak{m}_{X,D_i}^2$ and (a representative of) $g_i f_i \in \mathfrak{m}_{X,D_i}$ is a generator since (a representative of) g_i is a unit in $\mathcal{O}_{X,D_i}$ and f_i is a generator of $\mathfrak{m}_{X,D_i}$). Hence, by Theorem B.18, the intersection product of $\langle g_1 \rangle \otimes \overline{f_1}^*$ with $\langle g_2 \rangle \otimes \overline{f_2}^*$ is the sum over $\overline{\{x\}}$ irreducible component of $D_1 \cap D_2$ of $((m_x)_\epsilon \langle v_x \rangle) \otimes (\overline{\pi_x}^* \otimes \overline{g_1 f_1}^*)$ (over the point x), where π_x is a uniformizing parameter for $\mathcal{O}_{X,x}/(g_1 f_1)$, v_x is a unit in $\mathcal{O}_{X,x}/(g_1 f_1)$, $m_x \in \mathbb{Z}$ and $g_2 f_2 = v_x \pi_x^{m_x} \in \mathcal{O}_{X,x}/(g_1 f_1)$. Note that since g_1 is a unit in $\mathcal{O}_{X,D_1}$ hence in $\mathcal{O}_{X,x}$, the ideal $(g_1 f_1)$ is equal to the ideal (f_1) in $\mathcal{O}_{X,x}$. Note that since g_2 is a unit in $\mathcal{O}_{X,D_2}$ hence in $\mathcal{O}_{X,x}$, $u_x := g_2^{-1} v_x$ is a unit in $\mathcal{O}_{X,x}$. Finally note that $((m_x)_\epsilon \langle g_2 u_x \rangle) \otimes (\overline{\pi_x}^* \otimes \overline{g_1 f_1}^*) = ((m_x)_\epsilon \langle g_1 g_2 u_x \rangle) \otimes (\overline{\pi_x}^* \otimes \overline{f_1}^*)$ to get the result. ☐

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